

RELATIVE CURRENTS

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ABSTRACT. In this paper we define currents relative to a free factor system. We prove that a fully irreducible outer automorphism relative to a free factor system acts with uniform north-south dynamics on a subspace of the space of projective relative currents.

1. INTRODUCTION

The study of the *outer automorphism group* $\text{Out}(\mathbb{F})$ of a free group \mathbb{F} of rank n is highly motivated by the mapping class group $\text{MCG}(\Sigma)$ of a surface Σ . The theory of $\text{MCG}(\Sigma)$ has benefited greatly from its action on the curve complex $\mathcal{C}(\Sigma)$ which was proved to be hyperbolic in [MM99]. Analogously, $\text{Out}(\mathbb{F})$ acts on the free factor complex \mathcal{FF}_n which was proved to be hyperbolic in [BF14]. But sometimes the parallels between the two theories are not straightforward. For instance, consider a mapping class group element acting on $\mathcal{C}(\Sigma)$ with a fixed point, that is, it fixes a curve α on Σ . We can then look at its action on the curve complex of the subsurface given by the complement of α . On the other hand, consider an outer automorphism which fixes a free factor A in \mathcal{FF}_n . Since the complement of A in \mathbb{F} is not well defined we cannot pass to the free factor complex of a free group of lower rank. In [HM14], Handel and Mosher define a *free factor complex relative to a free factor system* (also called *relative free factor complex*) which is an $\text{Out}(\mathbb{F})$ -analog of the curve complex of a subsurface. In the same paper they also prove hyperbolicity of the relative free factor complex for a *non-exceptional* free factor system.

In this paper we develop the machinery of *currents relative to a free factor system*. This machinery is then used in [Gup16] to classify the outer automorphisms that act with positive translation length on a relative free factor complex.

In [Bon88], Bonahon first defined a space of *geodesic currents* for surfaces such that it contains the set of simple closed curves as a dense set. He studied the embedding of Teichmüller space in the space of geodesic currents and recovered Thurston's compactification of Teichmüller space. Currents for free groups were first studied by Reiner Martin [Mar95] in his thesis. Analogous to geodesic currents, the space of currents for \mathbb{F} contains the set of conjugacy classes of elements of \mathbb{F} as a dense set. Currents for free groups have also been studied in [Kap05], [Kap06], [KL09].

Let \mathcal{A} be a free factor system of \mathbb{F} . In this paper we define a space of *currents relative to \mathcal{A}* (also called *relative currents*) such that it contains the conjugacy classes of elements of \mathbb{F} which are not contained in \mathcal{A} as a dense set. Let $\partial^2\mathbb{F}$ be the space of flip-invariant bi-infinite geodesics in a Cayley graph of \mathbb{F} . A *relative current* is a non-negative, additive, \mathbb{F} -invariant and flip-invariant function defined on the set of compact open sets of a subspace \mathbf{Y} of $\partial^2\mathbb{F}$ which depends on \mathcal{A} . See Section 3.2 for details. The subspace \mathbf{Y} is defined in such a way that the action of \mathbb{F} on \mathbf{Y} is cocompact.

We show that the space of projectivized relative currents $\mathbb{PRC}(\mathcal{A})$ is compact. In order to show that the set of certain conjugacy classes of elements of \mathbb{F} is dense in $\mathbb{PRC}(\mathcal{A})$, we extend a relative current to a *signed measured current* which is in fact non-negative on words of bounded length.

Our main result is a generalization of a theorem in [Mar95] (see also [Uya14]) which says that a fully irreducible outer automorphism acts with uniform north-south dynamics on a subspace of the

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space of projectivized currents. A fully irreducible outer automorphism that is a pseudo-Anosov on a surface with one boundary component fixes the conjugacy class of the boundary curve. Therefore in [Mar95], the space of projectivized currents is replaced by a subspace which is the closure of primitive conjugacy classes. For the same reason we pass to a similarly defined subspace $\mathcal{MRC}(\mathcal{A})$ of $\mathbb{P}\mathcal{RC}(\mathcal{A})$.

Let $\text{Out}(\mathbb{F}, \mathcal{A})$ be the subgroup of $\text{Out}(\mathbb{F})$ containing outer automorphisms that fix \mathcal{A} . After passing to a finite index subgroup of $\text{Out}(\mathbb{F}, \mathcal{A})$ we can assume that each free factor $[A]$ in \mathcal{A} is invariant under elements of $\text{Out}(\mathbb{F}, \mathcal{A})$. An outer automorphism $\Phi \in \text{Out}(\mathbb{F}, \mathcal{A})$ is *fully irreducible relative to \mathcal{A}* if no power of Φ fixes a proper free factor system of \mathbb{F} properly containing \mathcal{A} . Let $\zeta(\mathcal{A})$ be the sum of the number of free factors in \mathcal{A} and the rank of a cofactor of \mathcal{A} .

Theorem A. *Let \mathcal{A} be a non-trivial free factor system of \mathbb{F} with $\zeta(\mathcal{A}) \geq 3$. Let $\Phi \in \text{Out}(\mathbb{F}, \mathcal{A})$ be fully irreducible relative to \mathcal{A} . Then Φ acts with uniform north-south dynamics on $\mathcal{MRC}(\mathcal{A})$, that is, there are only two fixed points $[\eta_\Phi^+]$ and $[\eta_\Phi^-]$ and any compact set that does not contain $[\eta_\Phi^-](\eta_\Phi^+)$ uniformly converges to $[\eta_\Phi^+](\eta_\Phi^-)$ under $\Phi(\Phi^{-1})$ iterates respectively.*

We use substitution dynamics techniques to understand the stable current η_Φ^+ and the unstable current η_Φ^- . In the absolute case, the transition matrix of a fully irreducible outer automorphism is primitive so Perron-Frobenius theory and the techniques in [Que87, Chapter 5] can be used to define the stable and unstable current. Since the transition matrix of a relative fully irreducible outer automorphism is not primitive and the complement of \mathcal{A} is not well-defined we have to do some work to compute the frequencies of mixed words in the limit. Also some care is required to view an outer automorphism as a substitution due to presence of exceptional paths in a completely split train representative. See Section 8.2 for details. We then study the legal and illegal turn structure of a conjugacy class under iteration by Φ .

Our main application of Theorem A is the following result about loxodromic elements in the relative free factor complex $\mathcal{FF}(\mathbb{F}, \mathcal{A})$.

Theorem B ([Gup16]). *Let \mathcal{A} be a non-exceptional free factor system of \mathbb{F} . Let $\Phi \in \text{Out}(\mathbb{F}, \mathcal{A})$ be fully irreducible relative to \mathcal{A} . Then Φ acts loxodromically on $\mathcal{FF}(\mathbb{F}, \mathcal{A})$.*

Plan of the paper. In Section 2 we review basic definitions. In Section 3 we define relative currents. In Section 4 we state the main proposition (from Section 8.2) about calculating frequencies of paths in a completely split train track representative of a relative fully irreducible outer automorphism. In Section 5 the stable and unstable relative currents are defined. In Section 6 we collect some lemmas about the legal and illegal turn structure of a conjugacy class under iterates by a relative fully irreducible outer automorphism. We conclude with the proof of the main theorem in Section 7. In the appendix we talk about substitution dynamics and extending relative currents to signed measured currents.

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2. PRELIMINARIES

2.1. Marked graphs and topological representatives. We review some terminology from [BH92]. Identify \mathbb{F} with $\pi_1(\mathcal{R}, *)$ where \mathcal{R} is a rose with n petals and n is the rank of \mathbb{F} . A *marked graph* G is a graph of rank n , all of whose vertices have valence at least two, equipped with a homotopy equivalence $m : \mathcal{R} \rightarrow G$ called a marking. The marking determines an identification of \mathbb{F} with $\pi_1(G, m(*))$.

A homotopy equivalence $\phi : G \rightarrow G$ induces an outer automorphism of $\pi_1(G)$ and hence an element Φ of $\text{Out}(\mathbb{F})$. If ϕ sends vertices to vertices and the restriction of ϕ to edges is an immersion then we say that ϕ is a *topological representative* of Φ .

A *filtration* for a topological representative $\phi : G \rightarrow G$ is an increasing sequence of (not necessarily connected) ϕ -invariant subgraphs $\emptyset = G_0 \subset G_1 \subset \dots \subset G_K = G$. The closure of $G_r \setminus G_{r-1}$, denoted H_r is a subgraph called the r^{th} -*stratum*.

Let γ be a reduced path in G . Then $\phi(\gamma)$ is the image of γ under the map ϕ . We will denote the tightened image of $\phi(\gamma)$ by $[\phi(\gamma)]$.

A path σ is a *Nielsen path* if the end points of σ are fixed and $\phi(\sigma)$ is homotopic relative end points to σ . A Nielsen path is *indivisible* if it does not decompose as a concatenation of non-trivial Nielsen subpaths. A path σ is a *pre-Nielsen path* if $\phi^k(\sigma)$ is a Nielsen path for some $k > 0$.

2.2. Train track maps.

We recall some more definitions from [BH92].

Let G be a marked graph. A *turn* in G is a pair of oriented edges of G originating at a common vertex. A turn is non-degenerate if the edges are distinct, it is degenerate otherwise. A *turn* (e_1, e_2) is *contained in a filtration element* G_r if both e_1 and e_2 are contained in G_r . If γ is an edge path given by $e_1 e_2 \dots e_{m-1} e_m$ then we say that γ *contains the turn* (\bar{e}_{i-1}, e_i) where \bar{e}_i denotes opposite orientation.

For $\phi : G \rightarrow G$ a topological representative and an edge e in G , we set $T\phi(e)$ equal to the first oriented edge of the edge path $\phi(e)$. Given a turn (e_1, e_2) define $T\phi(e_1, e_2) = (T\phi(e_1), T\phi(e_2))$. We say a turn is *illegal* if under some iterate of $T\phi$ the turn maps to a degenerate turn, it is *legal* otherwise. A path γ is called *r-legal* if all of its illegal turns are contained in G_{r-1} .

We associate a matrix called *transition matrix*, denoted M_r , to each stratum H_r . The ij^{th} entry of M_r is the number of occurrences of the i^{th} edge of H_r in either orientation in the image of the j^{th} edge under ϕ . A non-negative matrix M is called *irreducible* if for every i, j there exists $k(i, j) > 0$ such that ij^{th} entry of M^k is positive. A matrix is called *primitive* or *aperiodic* if there exists $k > 0$ such that M^k is positive. A stratum is called *zero stratum* if the transition matrix is the zero matrix. If M_r is irreducible then its Perron-Frobenius eigenvalue λ_r is greater than equal to 1. We say such a stratum is *exponentially growing (EG)* if $\lambda_r > 1$, it is called *non-exponentially growing (NEG)* otherwise.

A topological representative $\phi : G \rightarrow G$ of a free group outer automorphism Φ is a *relative train track map* with respect to a filtration $\emptyset = G_0 \subset G_1 \subset \dots \subset G_K = G$ if G has no valence one vertices, if each non-zero stratum has an irreducible matrix and if each exponentially growing stratum satisfies the following conditions:

- If E is an edge in H_r then the first and the last edges in $\phi(E)$ are also in H_r .
- If $\gamma \in G_{r-1}$ is a non-trivial path with endpoints in $H_r \cap G_{r-1}$ then $[\phi(\gamma)]$ is non-trivial with endpoints in $H_r \cap G_{r-1}$.
- For each r -legal path $\beta \subset H_r$, $[\phi(\beta)]$ is r -legal.

A reduced path $\sigma \subset G$ has *height* r if the highest stratum it crosses is G_r .

2.3. Completely split train track maps. In [FH11], Feighn and Handel defined completely split train track maps for outer automorphisms, which are better versions of relative train track maps. Instead of giving a full definition we list some facts about them which are used in this paper and then describe a complete splitting. Let $\phi : G \rightarrow G$ be a completely split train track map. The following facts proved in different papers can be found in [HM13, Section 1.5.2].

Facts 2.1. (1) Every periodic Nielsen path has period one.

- (2) If H_r is an exponentially growing stratum then there is at most one indivisible Nielsen path (INP) in G_r that intersects H_r nontrivially.
- (3) If H_r is an EG stratum and if ρ_r is an INP of height r , then ρ_r crosses each edge of H_r at least once, the initial oriented edges of ρ_r and $\bar{\rho}_r$ are distinct oriented edges of H_r , and:
 - (a) ρ_r is not closed iff it crosses some edge of H_r exactly once and in this case:

- (i) at least one end point of ρ is not in G_{r-1} .
- (ii) There does not exist a height r fixed conjugacy class.
- (b) ρ_r is closed iff it crosses each edge of H_r exactly twice, and in this case:
 - (i) the endpoint of ρ_r is not in G_{r-1} .
 - (ii) the only height r fixed conjugacy classes are those represented by ρ_r , its inverse and their iterates.

If H_r is an EG stratum then a non-trivial path in G_{r-1} with end points in $H_r \cap G_{r-1}$ is called a *connecting path*. If an NEG stratum H_i is a single edge e_i such that $\phi(e_i) = e_i u_i$ for a non-trivial closed Nielsen path u_i then e_i is called a *linear edge*. Let $u_i = w_i^{d_i}$ for some $d_i \neq 0$ where w_i is root-free. If e_i and e_j are distinct linear edges such that $\phi(e_i) = e_i w_i^{d_i}$ and $\phi(e_j) = e_j w_j^{d_j}$ where $d_i \neq d_j$ and $d_i, d_j > 0$ then a path of the form $e_i w_i^p \bar{e}_j$ where $p \in \mathbb{Z}$ is called an *exceptional path*.

A decomposition of a path or a circuit σ into subpaths is called a *splitting* if one can tighten the image of σ under ϕ by tightening the image of each subpath. In other words, there is no cancellation between images of two adjacent subpaths in the decomposition of σ .

Let e be an edge in an irreducible stratum H_r and let $k > 0$. A maximal subpath σ of tightened $\phi^k(e)$ in a zero stratum H_i is said to be *r-taken*. A non-trivial path or circuit in G is said to be *completely split* if it has a splitting into subpaths each of which is either a simple edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path or a connecting path in a zero stratum H_i that is taken and is maximal in H_i .

A relative train track map is *completely split* if for every edge e in each irreducible stratum $\phi(e)$ is completely split and if σ is a taken connecting path in a zero stratum then $[\phi(\sigma)]$ is completely split.

2.4. Free factor system. A free factor system of \mathbb{F} is a finite collection of proper free factors of \mathbb{F} of the form $\mathcal{A} = \{[A_1], \dots, [A_k]\}$, $k \geq 0$ such that there exists a free factorization $\mathbb{F} = A_1 * \dots * A_k * F_N$, where $[\cdot]$ denotes the conjugacy class of a subgroup. We refer to the free factor F_N as the *cofactor* of \mathcal{A} keeping in mind that it is not unique, even up to conjugacy. There is a partial ordering \sqsubset on the set of free factor systems given as follows: $\mathcal{A} \sqsubset \mathcal{A}'$ if for every $[A_i] \in \mathcal{A}$ there exists $[A'_j] \in \mathcal{A}'$ such that $A_i \subset A'_j$ up to conjugation. The free factor systems \emptyset and $\{[\mathbb{F}]\}$ are called *trivial free factor systems*. We define $\text{rank}(\mathcal{A})$ to be the sum of the ranks of the free factors in \mathcal{A} and let $\zeta(\mathcal{A}) = k + N$.

The main geometric example of a free factor system is as follows: suppose G is a marked graph and K is a subgraph whose non-contractible connected components are denoted C_1, \dots, C_k . Let $[A_i]$ be the conjugacy class of a free factor of \mathbb{F} determined by $\pi_1(C_i)$. Then $\mathcal{A} = \{[A_1], \dots, [A_k]\}$ is a free factor system. We say \mathcal{A} is *realized by* K and denote it by $\mathcal{F}(K)$.

2.5. Relative free factor complex. Let \mathcal{A} be a non-trivial free factor system of \mathbb{F} . In [HM14] the complex of free factor systems of \mathbb{F} relative to \mathcal{A} , denoted $\mathcal{FF}(\mathbb{F}; \mathcal{A})$, is defined to be the geometric realization of the partial ordering \sqsubset restricted to the set of non-trivial free factor systems \mathcal{B} of \mathbb{F} such that $\mathcal{A} \sqsubset \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$. The *exceptional* free factor systems are certain ones for which $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is either empty or zero-dimensional. They can be enumerated as follows:

- $\mathcal{A} = \{[A_1], [A_2]\}$ with $\mathbb{F} = A_1 * A_2$. In this case $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is empty.
- $\mathcal{A} = \{[A]\}$ with $\mathbb{F} = A * \mathbb{Z}$. In this case $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is 0-dimensional.
- $\mathcal{A} = \{[A_1], [A_2], [A_3]\}$ with $\mathbb{F} = A_1 * A_2 * A_3$. In this case $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is also 0-dimensional.

Theorem 2.2 ([HM14]). *For any non-exceptional free factor system \mathcal{A} of \mathbb{F} , the complex $\mathcal{FF}(\mathbb{F}, \mathcal{A})$ is positive dimensional, connected and hyperbolic.*

2.6. Fully irreducible relative to \mathcal{A} . Let \mathcal{A} be a non-trivial free factor system. An outer automorphism $\Phi \in \text{Out}(\mathbb{F}, \mathcal{A})$ is called *irreducible relative to \mathcal{A}* if there is no proper Φ -invariant free factor system that properly contains \mathcal{A} . If every power of Φ is irreducible relative to \mathcal{A} then we say that Φ is *fully irreducible relative to \mathcal{A}* (or relative fully irreducible).

Let $\Phi \in \text{Out}(\mathbb{F}, \mathcal{A})$. Then by Lemma 2.6.7 [BFH00] there exists a relative train track map for Φ , denoted $\phi : G \rightarrow G$, and filtration $\emptyset = G_0 \subset G_1 \subset \dots \subset G_r = G$ such that $\mathcal{A} = \mathcal{F}(G_s)$ for some filtration element G_s . If Φ is fully irreducible relative to \mathcal{A} then $\mathcal{A} = \mathcal{F}(G_{r-1})$ and the top stratum H_r is an EG stratum with Perron-Frobenius eigenvalue $\lambda_\Phi > 1$.

2.7. Bounded cancellation constant and critical length.

Lemma 2.3 ([Coo87]). *Let G be a marked metric graph and let $\phi : G \rightarrow G$ be a homotopy equivalence. There exists a constant $BCC(\phi)$, called the bounded cancellation constant, depending only on ϕ such that for any path ρ in G obtained by concatenating two paths α, β , we have*

$$L(\phi(\rho)) \geq L(\phi(\alpha)) + L(\phi(\beta)) - BCC(\phi)$$

where L is the length function on G .

Let $BCC(\phi)$ be the bounded cancellation constant for $\phi : G \rightarrow G$, a relative train track representative of a relative fully irreducible outer automorphism Φ with top EG stratum H_r . Assume G has a metric such that edges in H_r get stretched by the PF eigenvalue λ_Φ under ϕ and edges in G_{r-1} have length one. Let l_r denote the r -length. Let α, β, γ be r -legal paths in G . Let $\alpha.\beta.\gamma$ be the path obtained by concatenating these r -legal paths. The only r -illegal turns possibly occur at the ends of the segments of β . Thus if $\lambda_\Phi l_r(\beta) - 2BCC(\phi) > l_r(\beta)$ then iterations and tightening of $\alpha.\beta.\gamma$ will produce paths with r -length of legal segments corresponding to β going to infinity. We call $\frac{2BCC(\phi)}{\lambda_\Phi - 1}$ the *critical length* for ϕ .

3. RELATIVE CURRENTS

The goal of this section is to define a space of currents relative to a free factor system.

3.1. Boundary of \mathbb{F} . Given \mathbb{F} and a fixed basis \mathfrak{B} of \mathbb{F} let $\text{Cay}(\mathbb{F}, \mathfrak{B})$ be the Cayley graph of \mathbb{F} with respect to \mathfrak{B} . The space of ends of the Cayley graph is called the *boundary* of \mathbb{F} , denoted by $\partial\mathbb{F}$. It is homeomorphic to the Cantor set. A one-sided cylinder set determined by a finite path γ starting at the base point is the set of all rays starting at the base point that cross γ . Such cylinder sets form a basis for the topology on $\partial\mathbb{F}$ and are in fact both open and closed.

Let Δ denote the diagonal in $\partial\mathbb{F} \times \partial\mathbb{F}$. Let $\partial^2\mathbb{F} := (\partial\mathbb{F} \times \partial\mathbb{F} - \Delta)/\mathbb{Z}_2$ be the space of flip-invariant bi-infinite lines in $\text{Cay}(\mathbb{F}, \mathfrak{B})$. This space is also called the *double boundary* of \mathbb{F} . The space $\partial^2\mathbb{F}$ gets product topology from $\partial\mathbb{F}$ and is Hausdorff. Finite paths γ in $\text{Cay}(\mathbb{F}, \mathfrak{B})$ determine two-sided cylinder sets, denoted $C(\gamma)$, which form a basis for the topology. Two-sided cylinder sets are open and compact and hence closed. Compact open sets are given by finite disjoint union of cylinder sets. Also $\partial^2\mathbb{F}$ is locally compact but not compact. The action of \mathbb{F} on $\partial^2\mathbb{F}$ is cocompact.

Let $\mathcal{A} = \{[A_1], \dots, [A_k]\}$, $k > 0$, be a free factor system such that $\zeta(\mathcal{A}) \geq 3$.

Definition 3.1 (Relative basis). Let $\mathfrak{B}_\mathcal{A}$ be a basis of \mathbb{F} such that a basis of \mathcal{A} is a subset of $\mathfrak{B}_\mathcal{A}$. Specifically,

$$\mathfrak{B}_\mathcal{A} = \{a_{11}, \dots, a_{11_s}, \dots, a_{i1}, \dots, a_{ii_s}, \dots, a_{k1}, \dots, a_{kk_s}, b_1, \dots, b_p\}$$

where $a_{ij} \in A_i$ and $b_i \notin \mathcal{A}$ for any $[A] \in \mathcal{A}$. Let $\sum_{i=1}^k i_s =: s$. We define a set $B_\mathcal{A}$ to be the union of the set of all words $a_{ij}^\pm a_{kl}^\pm$ of length two such that $i \neq j$ and the set of all b_i . Note that if $\text{rank}(\mathcal{A}) = \text{rank}(\mathbb{F})$ then the set of b_i is empty. We call $\mathfrak{B}_\mathcal{A}$ a *relative basis* of \mathbb{F} .

Definition 3.2 (Double boundary of \mathcal{A}). Given a free factor A we define $\partial^2 A$ to be the set of bi-infinite geodesics in $\partial^2\mathbb{F}$ which are lifts of conjugacy classes of elements in A . We then define the double boundary of \mathcal{A} as $\partial^2\mathcal{A} := \bigsqcup_{i=1}^k \partial^2 A_i$.

Let $\mathbb{F} \setminus \mathcal{A}$ be the set of all conjugacy classes of elements in \mathbb{F} that are not contained in a free factor in \mathcal{A} . Note that an element of $\mathbb{F} \setminus \mathcal{A}$ can be contained in the free product of distinct free factors of \mathcal{A} .

Let $\mathbf{Y} = \partial^2 \mathbb{F} \setminus \partial^2 \mathcal{A}$. It inherits the subspace topology from $\partial^2 \mathbb{F}$. It can also be given a topology where cylinder sets in \mathbf{Y} determined by finite paths that cross at least one word in $B_{\mathcal{A}}$ form a basis for the topology. The two topologies are in fact equivalent.

Lemma 3.3. *\mathbf{Y} is locally compact.*

Proof. A space is locally compact if every point has a compact neighborhood. Let x be an element of \mathbf{Y} . Take a finite subpath of x that cannot be written as a string of words contained in a single $[A] \in \mathcal{A}$ and consider the cylinder set determined by that path. Then this cylinder set is a compact open set in \mathbf{Y} containing x . \square

Lemma 3.4. *The action of \mathbb{F} on \mathbf{Y} is cocompact.*

Proof. Consider a compact set $C \subset \text{Cay}(\mathbb{F}, \mathfrak{B}_{\mathcal{A}})$ given by a finite union of cylinder sets determined by all paths with one end point at the origin and with label a word in $B_{\mathcal{A}}$. For every bi-infinite geodesic γ in \mathbf{Y} there is a $g \in \mathbb{F}$ such that $g \cdot \gamma$ crosses a path starting at the origin determined by a word in $B_{\mathcal{A}}$. \square

3.2. Definition of relative current. We first recall the definition of a measured current as defined in [Mar95]. A *measured current* is an additive, non-negative, \mathbb{F} -invariant and flip-invariant function on the set of compact open sets in $\partial^2 \mathbb{F}$. It is uniquely determined by its values on the cylinder sets determined by conjugacy classes in \mathbb{F} .

Let $\mathcal{C}(\mathbf{Y})$ be the collection of compact open sets in \mathbf{Y} . A *relative current* is an additive, non-negative, \mathbb{F} -invariant and flip-invariant function on $\mathcal{C}(\mathbf{Y})$. Let $\mathcal{RC}(\mathcal{A})$ denote the space of relative currents. A subbasis for the topology of $\mathcal{RC}(\mathcal{A})$ is given by the sets $\{\eta \in \mathcal{RC}(\mathcal{A}) : |\eta(C) - \eta_0(C)| \leq \epsilon\}$ where $\eta_0 \in \mathcal{RC}(\mathcal{A})$, $C \in \mathcal{C}(\mathbf{Y})$ and $\epsilon > 0$.

$\text{Out}(\mathbb{F}, \mathcal{A})$ acts on $\mathcal{RC}(\mathcal{A})$ as follows: let $\eta \in \mathcal{RC}(\mathcal{A})$, $\Psi \in \text{Out}(\mathbb{F}, \mathcal{A})$ and let $C \in \mathcal{C}(\mathbf{Y})$. Then

$$\Psi.\eta(C) = \eta(\Psi^{-1}(C)).$$

A relative current can also be defined as an \mathbb{F} -invariant, locally finite, inner regular measure (called Radon measure) on the σ -algebra of Borel sets of \mathbf{Y} .

Lemma 3.5. *A non-negative, additive function on $\mathcal{C}(\mathbf{Y})$ corresponds to a Radon measure on the Borel σ -algebra of \mathbf{Y} .*

Proof. Given a non-negative, additive function η on $\mathcal{C}(\mathbf{Y})$ we define an outer measure $\eta^* : 2^{\mathbf{Y}} \rightarrow [0, \infty]$ as follows: for $A \in 2^{\mathbf{Y}}$

$$\eta^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \eta(C_i) : A \subseteq \bigcup_{i=1}^{\infty} C_i \text{ where } C_i \in \mathcal{C}(\mathbf{Y}) \text{ is a cylinder set} \right\}.$$

We have $\eta^*(C) = \eta(C)$ for $C \in \mathcal{C}(\mathbf{Y})$ because every cover of a compact set has a finite subcover and then we can use additivity of η . A cylinder set C in $\mathcal{C}(\mathbf{Y})$ is outer measurable, that is, for every $A \in 2^{\mathbf{Y}}$ we have $\eta^*(A) = \eta^*(A \cap C^c) + \eta^*(A \cap C)$. An outer measure is a measure on the σ -algebra of outer measurable sets. But here the σ -algebra of outer measurable sets is equal to the σ -algebra of Borel sets. Therefore this outer measure η^* is a measure on the Borel σ -algebra of \mathbf{Y} . The space \mathbf{Y} is locally compact and Hausdorff and every open set in \mathbf{Y} is σ -compact, that is, a countable union of compact sets. Also η^* is a non-negative Borel measure on \mathbf{Y} such that it is finite on compact sets. Therefore η^* is a regular measure. \square

Thus we can give the space of relative currents a weak-* topology, that is, $\eta_n \rightarrow \eta$ in $\mathcal{RC}(\mathcal{A})$ iff $\int_{\mathbf{Y}} f d\eta_n \rightarrow \int_{\mathbf{Y}} f d\eta$ for all compactly supported functions f on \mathbf{Y} . Since \mathbf{Y} is a locally compact space, by a result in [Bou65, Chapter III, Section 1] $\mathcal{RC}(\mathcal{A})$ is complete.

3.3. Coordinates for $\mathcal{RC}(\mathcal{A})$. Fix a relative basis $\mathfrak{B}_{\mathcal{A}}$ of \mathbb{F} . Given $w \neq 1 \in \mathbb{F}$ consider the unique oriented path, denoted γ_w , determined by w starting at the base point and let $C(w) := C(\gamma_w)$. Note that this cylinder set contains *unoriented* bi-infinite geodesics that cross γ_w . For $w \in \mathbb{F} \setminus \mathcal{A}$ we have $C(w) \subset \mathcal{C}(\mathbf{Y})$. Orbits of cylinder sets of the form $C(w)$ under deck transformations cover \mathbf{Y} . We denote η applied to $C(w)$ by $\eta(w)$.

- Let $v \in \mathbb{F}$. Then $v \cdot C(w)$ is the set of all bi-infinite geodesics that cross an edge path labeled by w starting at the vertex labeled v in the Cayley graph. By \mathbb{F} -invariance of a relative current we have that $\eta(C(w)) = \eta(v \cdot C(w))$. Thus we can work just with the cylinder sets determined by finite paths starting at the base point. Since every compact open set is a finite disjoint union of cylinder sets, a relative current is uniquely determined by its values on $\mathbb{F} \setminus \mathcal{A}$.
- Since a relative current is uniquely determined by its values on $\mathbb{F} \setminus \mathcal{A}$ we have that a sequence of relative currents η_n converges to η iff $\eta_n(w) \rightarrow \eta(w)$ for all $w \in \mathbb{F} \setminus \mathcal{A}$.
- For any finite path γ in $\text{Cay}(\mathbb{F}, \mathfrak{B}_{\mathcal{A}})$ we have $C(\gamma) = C(\bar{\gamma})$, where $\bar{\gamma}$ denotes the opposite orientation on γ . If w and γ_w are as above then $C(w) = C(\gamma_w) = C(\bar{\gamma}_w) = w \cdot C(\bar{w})$. Thus $\eta(w) = \eta(\bar{w})$.
- Let $w = e_0 e_1 \dots e_l \in \mathbb{F} \setminus \mathcal{A}$ where each $e_i \in \mathfrak{B}_{\mathcal{A}}$. Then $C(w) = \cup C(we)$ where the union is taken over all basis elements in $\mathfrak{B}_{\mathcal{A}}$ except $e = \bar{e}_l$. Here \bar{e} denotes the inverse of e . Also $C(w) = \cup \bar{e} \cdot C(ew)$ where e is any basis element other than \bar{e}_0 . Thus additivity of a relative current can be stated as

$$\eta(w) = \sum_{e \neq \bar{e}_l} \eta(we) \quad \text{or} \quad \eta(w) = \sum_{e \neq \bar{e}_0} \eta(ew).$$

For example, let $\mathbb{F} = \langle a, b \rangle$ and $\mathcal{A} = \{[\langle a \rangle]\}$, we have

$$\begin{aligned} \eta(b) &= \eta(ba) + \eta(bb) + \eta(b\bar{a}), \\ \eta(b) &= \eta(ab) + \eta(bb) + \eta(\bar{a}b) \end{aligned}$$

- Let $v, w \in \mathbb{F} \setminus \mathcal{A}$ be such that v is a subword of w . Then $\eta(w) \leq \eta(v)$.

Example 3.6 (Relative current). Consider a conjugacy class $\alpha \in \mathbb{F} \setminus \mathcal{A}$ such that α is not a power of any other conjugacy class in \mathbb{F} . Then $\eta_{\alpha}(w)$ is the number of occurrences of w in the cyclic words α and $\bar{\alpha}$. Equivalently, one can also count the number of lifts of α that cross the path γ_w in the Cayley graph. We call such currents and their multiples *rational relative currents*. For example, let $\mathbb{F} = \langle a, b \rangle$, $\mathcal{A} = \{[\langle a \rangle]\}$ and let $\alpha = abaab$. Then $\eta_{\alpha}(b) = 2$, $\eta_{\alpha}(ba) = 2$, $\eta_{\alpha}(abab) = 1$.

Given $w \in \mathbb{F}$, a *length k extension* of w is a word $w' = wx_1 \dots x_k$ where $x_i \in \mathfrak{B}_{\mathcal{A}}$, $x_i \neq \overline{x_{i+1}}$ and x_1 is not the inverse of the last letter of w .

Lemma 3.7. *Any non-negative function η on $\mathbb{F} \setminus \mathcal{A}$ invariant under inversion and the action of \mathbb{F} , and satisfying the condition*

$$\eta(w) = \sum_{\substack{\text{length one} \\ \text{extension of } w}} \eta(v)$$

for all $w \in \mathbb{F} \setminus \mathcal{A}$ determines a relative current.

Proof. A set $C \in \mathcal{C}(\mathbf{Y})$ can be written as a disjoint union of cylinder sets $C(w_1), \dots, C(w_k)$ with $w_i \in \mathbb{F} \setminus \mathcal{A}$. Then define $\eta(C) := \sum_{i=1}^k \eta(w_i)$. The value $\eta(C)$ does not depend on the choice of w_i . Thus we have an additive non-negative function on $\mathcal{C}(\mathbf{Y})$ which is invariant under the action of \mathbb{F} . \square

3.4. Projectivized relative currents. We denote by $\mathbb{P}\mathcal{RC}(\mathcal{A})$ the space of projectivized relative currents. It has quotient topology induced from $\mathcal{RC}(\mathcal{A})$. A sequence of projective currents $[\eta_i]$ converges to $[\eta]$ in $\mathbb{P}\mathcal{RC}(\mathcal{A})$ iff there exist scaling constants a_i such that relative currents $a_i\eta_i$ converge to η in $\mathcal{RC}(\mathcal{A})$.

Example 3.8. Let $\mathbb{F} = \langle a, b \rangle$ and let $\mathcal{A} = \{[\langle a \rangle]\}$. Consider the sequence $\eta_{a^n b} \in \mathcal{RC}(\mathcal{A})$. This sequence converges to a relative current η_∞ which is given by $\eta_\infty(a^n b) = \eta_\infty(b a^n) = 1$ for all $n \geq 0$ and $\eta_\infty(w) = 0$ for all other $w \in \mathbb{F} \setminus \mathcal{A}$. Whereas in the space of measured currents $\mathcal{MC}(\mathbb{F})$ as defined in [Mar95], the sequence $\eta_{a^n b}/n$ converges to the current η_a .

Lemma 3.9. $\mathbb{P}\mathcal{RC}(\mathcal{A})$ is compact.

Proof. Consider a sequence of projective relative currents $[\eta_n]$. We have to show that it has a convergent subsequence. Fix a relative basis $\mathfrak{B}_{\mathcal{A}}$ and the associated set $B_{\mathcal{A}} = \{u_1, \dots, u_r\}$ (see Definition 3.1). Let η_n be a representative of $[\eta_n]$ normalized such that $\eta_n(u_i) \leq 1$ for all $u_i \in B_{\mathcal{A}}$ and $\eta_n(u_j) = 1$ for some $u_j \in B_{\mathcal{A}}$. We have $\eta_n(w) \leq \eta_n(u_i)$ where $w \in \mathbb{F} \setminus \mathcal{A}$ and crosses a path labeled $u_i \in B_{\mathcal{A}}$ in $\text{Cay}(\mathbb{F}, \mathfrak{B}_{\mathcal{A}})$. The bounded sequence $\{(\eta_n(u_1), \dots, \eta_n(u_r))\}_{n \in \mathbb{N}}$ has a subsequence that converges to a non-zero element of \mathbb{R}^r . For every $w \in \mathbb{F} \setminus \mathcal{A}$ $\{\eta_n(w)\}_{n \in \mathbb{N}}$ is a bounded sequence and hence has a convergent subsequence. Now we can do the diagonal argument to conclude that $\{(\eta_n(w))_{w \in \mathbb{F} \setminus \mathcal{A}}\}_{n \in \mathbb{N}}$ has a subsequence that converges to a non-zero element. Thus $\{[\eta_n]\}_{n \in \mathbb{N}}$ has a convergent subsequence in $\mathbb{P}\mathcal{RC}(\mathcal{A})$. \square

3.5. Density of rational relative currents.

Proposition 3.10. *The set of projectivized relative currents induced by conjugacy classes $\alpha \in \mathbb{F} \setminus \mathcal{A}$ are dense in $\mathbb{P}\mathcal{RC}(\mathcal{A})$.*

Let $\mathfrak{B}_{\mathcal{A}}$ be a relative basis of \mathbb{F} and let $|w|$ denote the word length of $w \in \mathbb{F}$ in the basis $\mathfrak{B}_{\mathcal{A}}$. The following lemma is the main step to prove density of rational currents in the absolute case. But it doesn't directly apply to the relative setting as explained below.

Lemma 3.11 ([Mar95, Lemma 15]). *Let η be a measured current and let $k \geq 2$. Let $P = 2n(2n-1)^{2n(2n-1)^{k-2}}$ be a constant. If $\eta(w_0) \geq P$ for some $w_0 \in \mathbb{F}$ with $|w_0| = k$ then there exists a conjugacy class $\alpha \in \mathbb{F}$ and the corresponding measured current η_α with $\eta(w) \geq \eta_\alpha(w)$ for all $w \in \mathbb{F}$ and $|w| \leq k$.*

The proof of the above lemma relies on finding cycles in a certain labeled directed graph associated to η defined as follows: vertices are given by words of length $k-1$ and edges are given by words of length k . A directed edge w joins vertex u to vertex v if u is the prefix of w and v is the suffix of w . An edge w is labeled by $\eta(w)$. Since η satisfies additivity laws for all words in \mathbb{F} , this graph satisfies Kirchhoff's law at each vertex which is crucial to find cycles (which correspond to α) in the graph. The same graph defined for a relative current η_0 does not satisfy Kirchhoff's law at vertices which correspond to words in \mathcal{A} because η_0 is not defined for words in \mathcal{A} .

A signed measured current on $\partial^2 \mathbb{F}$ is an \mathbb{F} -invariant and additive function on the set of compact open sets of $\partial^2 \mathbb{F}$. We now restate the above lemma for a signed measured current which is non-negative on words in \mathbb{F} of bounded length.

Lemma 3.12. *Let $k \geq 2$ and let η be a signed measured current such that $\eta(w) \geq 0$ for all $w \in \mathbb{F}$ with $|w| \leq k$. Let $P = 2n(2n-1)^{2n(2n-1)^{k-2}}$ be a constant. If $\eta(w_0) \geq P$ for some $w_0 \in \mathbb{F}$ with $|w_0| = k$ then there exists a conjugacy class $\alpha \in \mathbb{F}$ and the corresponding measured current η_α with $\eta(w) \geq \eta_\alpha(w)$ for all $w \in \mathbb{F}$ and $|w| \leq k$.*

For $\eta_0 \in \mathcal{RC}(\mathcal{A})$, let η be a signed measured current such that $\eta(w) = \eta_0(w)$ for $w \in \mathbb{F} \setminus \mathcal{A}$ and $\eta(w) \geq 0$ for all words $w \in \mathbb{F}$ with $|w| \leq k$. We call such an η a k -extension of η_0 .

Lemma 3.13. *Let η_0 be a relative current and let $k \geq 1$. Then there exists a signed measured current η which is a k -extension of η_0 .*

To prove the above lemma we start by defining η on words in \mathcal{A} of length one arbitrarily and then extend the current to length two words by satisfying the additivity property. One has to check that the constraints obtained from the additive property are consistent. A detailed proof is given in Section 8.1. Assuming the above lemma is true we now prove Proposition 3.10.

Proof of Proposition 3.10. We follow the same method of proof as in [Mar95]. Let η_0 be a relative current and let $k \geq 2$. Choose $R > 0$ such that $R\eta_0(w_0) \geq P$ for some $w_0 \in \mathbb{F} \setminus \mathcal{A}$ with $|w_0| = k$. Consider a signed measured current η which is a k -extension of η_0 . Then by Lemma 3.12 applied to $R\eta$ there exists an $\alpha_1 \in \mathbb{F}$ such that $R\eta(w) \geq \eta_{\alpha_1}(w)$ for all $w \in \mathbb{F}$ with $|w| \leq k$. If $R\eta(w) \leq \eta_{\alpha_1}(w) + P$ for all $w \in \mathbb{F}$ with $|w| \leq k$ then we stop, otherwise we again apply Lemma 3.12 to $R\eta - \eta_{\alpha_1}$ to obtain $\alpha_2 \in \mathbb{F}$ such that $R\eta(w) - \eta_{\alpha_1}(w) \geq \eta_{\alpha_2}(w)$ for all $w \in \mathbb{F}$ with $|w| \leq k$. By induction we will have $\sum \eta_{\alpha_i}(w) \leq R\eta(w) \leq \sum \eta_{\alpha_i}(w) + P$ for all words of length less than equal to k .

It is necessary that at least one of the $\alpha_i \in \mathbb{F} \setminus \mathcal{A}$. Indeed, if they were all in \mathcal{A} then $\sum \eta_{\alpha_i}(w_0) = 0$ which would mean $R\eta(w_0) \leq P$ which is a contradiction.

Now we have that

$$\left| \eta(w) - \frac{\sum \eta_{\alpha_i}(w)}{R} \right| \leq \frac{P}{R}$$

for all $w \in \mathbb{F}$ with $|w| \leq k$. For $w \in \mathbb{F} \setminus \mathcal{A}$ in fact

$$\left| \eta_0(w) - \frac{\sum_{\alpha_i \notin \mathcal{A}} \bar{\eta}_{\alpha_i}(w)}{R} \right| \leq \frac{P}{R}$$

where $\bar{\eta}_{\alpha_i}$ is the restriction of η_{α_i} to \mathbf{Y} .

Since R can be chosen arbitrarily large we can approximate relative currents by sums of rational relative currents for all $w \in \mathbb{F} \setminus \mathcal{A}$ with $|w| \leq k$. Now we can approximate $\sum_{\alpha_i \notin \mathcal{A}} \eta_{\alpha_i}$ by $\frac{1}{m} \eta_{\beta^m}$ where $\beta^m = w_1^m w_2^m \cdots w_l^m$ and w_i is in the conjugacy class of α_i . \square

3.6. Relative Whitehead Graph. We say a conjugacy class $w \in \mathbb{F} \setminus \mathcal{A}$ is \mathcal{A} -separable if it is contained in a proper free factor system containing \mathcal{A} . Topologically, w is \mathcal{A} -separable if there is an \mathbb{F} -tree T with set of vertex stabilizers given by \mathcal{A} such that an axis of w does not cross every orbit of edges. To detect when a conjugacy class is \mathcal{A} -separable we use Whitehead's algorithm and a theorem of Stallings [Sta99].

As defined in [Sta99], a collection \mathcal{C} of conjugacy classes in \mathbb{F} is *separable* if there exist free factors F, F' such that $\mathbb{F} = F * F'$ and each conjugacy class in \mathcal{C} is contained in either F or F' . Let $\alpha_i \in A_i$, $0 < i \leq k$, be a conjugacy class such that α_i is not contained in any proper free factor of A_i . We say α_i is *filling* in A_i .

Lemma 3.14. *A conjugacy class $w \in \mathbb{F}$ is \mathcal{A} -separable if and only if the collection of conjugacy classes $\mathcal{C} = \{w, \alpha_1, \dots, \alpha_k\}$ is separable.*

Proof. If \mathcal{C} is separable then there exist a decomposition $\mathbb{F} = F * F'$ such that each conjugacy class in \mathcal{C} is contained either in F or F' . Suppose $\alpha_i \in F$. Then we claim that A_i is contained in F up to conjugation. Suppose not. We have that $F \cap A_i \neq \emptyset$ up to conjugation. Also the intersection of two free factors is a free factor. So α_i is contained in a proper free factor of A_i , which is a contradiction. Thus $\{[F], [F']\}$ is a proper free factor system containing \mathcal{A} that contains the conjugacy class w .

On the other hand if w is contained in a proper free factor system \mathcal{D} containing \mathcal{A} then \mathcal{C} is separable. \square

Definition 3.15 (Whitehead Graph [Whi36]). Given a basis \mathfrak{B} of \mathbb{F} , the Whitehead graph of a collection of conjugacy classes, denoted $Wh(\mathcal{C})$, is defined as follows: the vertices are given by basis elements and their inverses. There is an edge connecting vertices x and y if $\bar{x}y$ is a subword of a conjugacy class in \mathcal{C} .

Theorem 3.16 ([Sta99, Theorem 4.2]). *Let \mathcal{C} be a collection of conjugacy classes in \mathbb{F} . If $Wh(\mathcal{C})$ is connected and \mathcal{C} is separable then there is a cut vertex in $Wh(\mathcal{C})$.*

Definition 3.17 (Relative Whitehead Graph). For each $[A_i] \in \mathcal{A}$ fix filling conjugacy classes $\alpha_i \in A_i$. The relative Whitehead graph of a conjugacy class $w \in \mathbb{F} \setminus \mathcal{A}$, denoted $Wh(w, \mathcal{A})$, is defined as the Whitehead graph of the collection $\{w, \alpha_1, \dots, \alpha_k\}$.

Note that even though we fix some filling conjugacy classes to define the relative Whitehead graph, detecting \mathcal{A} -separability of w is independent of them by Lemma 3.14.

3.7. A closed subspace of $\mathbb{PRC}(\mathcal{A})$. In the absolute case, when a fully irreducible outer automorphism Ψ is a pseudo-Anosov on a surface with one boundary component, the measured current corresponding to the boundary conjugacy class in the space of projectivized measured currents $\mathcal{MC}(\mathbb{F})$ is fixed under the action of Ψ . Thus in [Mar95], a closed subspace is considered which is the closure of all primitive conjugacy classes in $\mathcal{MC}(\mathbb{F})$. For the same reason we pass to a smaller closed $\text{Out}(\mathbb{F}, \mathcal{A})$ -invariant subspace of $\mathbb{PRC}(\mathcal{A})$. Let

$$\mathcal{MRC}(\mathcal{A}) = \overline{\{[\eta_w] \in \mathbb{PRC}(\mathcal{A}) \mid w \text{ is } \mathcal{A}\text{-separable}\}}$$

Lemma 3.18. *$[\eta_w] \in \mathbb{PRC}(\mathcal{A})$ is in $\mathcal{MRC}(\mathcal{A})$ iff w is \mathcal{A} -separable.*

Proof. Let's assume that w is not \mathcal{A} -separable. Then by Theorem 3.16 the relative Whitehead graph of w with respect to any relative basis is connected without a cut vertex. Consider a relative current η_v where $v \in \mathbb{F} \setminus \mathcal{A}$ such that $\eta_v(w^2) > 0$. This means that any relative Whitehead graph of v contains the Whitehead graph of w as a subgraph and hence is connected without cut vertices. By Theorem 3.16 and Lemma 3.14 this implies that v is not \mathcal{A} -separable. Thus $\eta_v(w^2) = 0$ for all \mathcal{A} -separable conjugacy classes v in $\mathbb{F} \setminus \mathcal{A}$, which in turn implies that $\eta(w) = 0$ for any $[\eta] \in \mathcal{MRC}(\mathcal{A})$. Since $\eta_w(w^2) > 0$ we have that $\eta_w \notin \mathcal{MRC}(\mathcal{A})$. \square

4. SUBSTITUTION DYNAMICS

In [Que87], a theory of substitution dynamics is developed for primitive substitutions to study their limiting behavior. This theory can be used to study a fully irreducible outer automorphism by viewing it as a substitution. In Section 8.2 of the Appendix, we develop a theory of substitution dynamics for a different class of substitutions in order to study outer automorphisms relative to a free factor system. Here we state the main proposition from Section 8.2 which allows us to calculate the frequencies of certain words in \mathbb{F} in the ray obtained by iterating some exponentially growing edge in a marked graph under an outer automorphism.

For γ and α two paths in a graph G , let (γ, α) be the number of occurrences of γ in α .

Proposition 4.1. *Let $\phi : G \rightarrow G$ be a completely split train track map. Let a be an edge in an EG stratum H_r such that $\phi(a)$ starts with a , and let $\rho_a := \lim_{n \rightarrow \infty} \phi^n(a)$. Let γ be a path in G_r that crosses H_r . Then*

$$\lim_{n \rightarrow \infty} \frac{(\gamma, \phi^n(a))}{\lambda^n} =: d_{\gamma, a}$$

exists and is non-negative. Here λ is the Perron-Frobenius eigenvalue of the aperiodic EG stratum H_r . If $b \in H_r$ is another edge then for every γ as above,

$$d_{\gamma, b} = \kappa d_{\gamma, a}$$

where κ is a constant with $\kappa = \kappa(a, b, \phi|_{H_r})$.

In general, it is possible that γ grows faster than λ due to the presence of subpaths in G_{r-1} that grow faster. The point of the above proposition is to ignore the contribution to the growth of γ from the lower stratum but still be able to compute frequencies of paths that cross H_r and are not necessarily completely contained in H_r .

5. STABLE AND UNSTABLE RELATIVE CURRENT

In this section we define the stable and unstable relative currents associated to a fully irreducible outer automorphism relative to \mathcal{A} . Before we state the general result let's look at some examples. The three examples that follow illustrate the cases when the growth in the stratum corresponding to \mathcal{A} is less than, greater than and equal to the growth in the top EG stratum.

Example 5.1. Let $F_3 = \langle a, b, c \rangle$. Let G be the rose on three petals labeled a, b and c . Consider an outer automorphism Φ given by a train track representative $\phi : G \rightarrow G$ where

$$\phi(a) = a, \phi(b) = bac, \phi(c) = cbac.$$

Let $\mathcal{A} = \{[\langle a \rangle]\}$. The transition matrix for ϕ is given by

$$M = \begin{matrix} & \begin{matrix} b & c & a \end{matrix} \\ \begin{matrix} b \\ c \\ a \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{matrix}.$$

Note that Φ is not fully irreducible relative to \mathcal{A} because the free factor system $\{[\langle b, ac \rangle], [\langle a \rangle]\}$ is Φ -invariant. But it is still instructive to understand the limiting behavior in this simple case.

Let $\rho_b = \lim_{n \rightarrow \infty} \phi^n(b)$ be a ray that is fixed by ϕ . We can view ϕ as a substitution ζ on the alphabet $\mathbb{A} = \{a, b, c\}$. Let \mathbb{A}_l be the set of words of length l on \mathbb{A} that appear in ρ_b . For example $\mathbb{A}_2 = \{ba, ca, cb, ac\}$. Note that the sets \mathbb{A}_l are independent of the specific choice b . We define a substitution ζ_l on \mathbb{A}_l as follows: let $w \in \mathbb{A}_l$ start with $x \in \mathbb{A}$. Then $\zeta_l(w)$ consists of the ordered list of the first $|\zeta(x)|$ subwords of length l of the word $\zeta(w)$. For example, $\zeta_2(ba) = ba \cdot ac \cdot ca$. Let M_l be the transition matrix of ζ_l and let \mathcal{B}_l be the transition matrix for ζ_l restricted to words in $\mathbb{F} \setminus \mathcal{A}$. We want to calculate the frequency of occurrences of words, which are not in \mathcal{A} , in ρ_b .

Let $w \in \mathbb{A}_l$ and let β be a word of length l that starts with b . Then

$$\lim_{n \rightarrow \infty} \frac{(w, \phi^n(b))}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{M_l^n(w, \beta)}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{\mathcal{B}_l^n(w, \beta)}{\lambda^n} =: d_{w, \beta}$$

Here λ is the PF-eigenvalue of the top EG stratum. For example in length one and two we have

$$\mathcal{B}_1 = \begin{matrix} & \begin{matrix} b & c \end{matrix} \\ \begin{matrix} b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{matrix}, \quad \mathcal{B}_2 = \begin{matrix} & \begin{matrix} ba & ca & cb & ac \end{matrix} \\ \begin{matrix} ba \\ ca \\ cb \\ ac \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{matrix}.$$

We take $\beta = b$ and $\beta = ba$ for length one and length two words respectively. Then

$$\begin{aligned} (b, \rho_b) &= \frac{(5 - \sqrt{5})}{10}, & (c, \rho_b) &= \frac{1}{\sqrt{5}} \\ (ac, \rho_b) &= \frac{1}{\sqrt{5}}, & (ba, \rho_b) &= \frac{(5 - \sqrt{5})}{10}, \\ (ca, \rho_b) &= \frac{(-5 + 3\sqrt{5})}{10}, & (cb, \rho_b) &= \frac{(5 - \sqrt{5})}{10} \end{aligned}$$

We get $(b, \rho_b) = (ba, \rho_b)$ and $(c, \rho_b) = (ca, \rho_b) + (cb, \rho_b)$ which indicates that additivity holds. One way to calculate the above numbers is to compute the Jordan decomposition of the matrix \mathcal{B}_l .

Example 5.2. Let $F_4 = \langle a, b, c, d \rangle$. Let G be the rose on four petals labeled a, b, c, d . Consider an outer automorphism Φ given by a train track representative $\phi : G \rightarrow G$ by

$$\phi(a) = abbab, \phi(b) = bababbab, \phi(c) = cad, \phi(d) = dcad.$$

Let $\mathcal{A} = \{[\langle a, b \rangle]\}$. The transition matrix for ϕ is given by

$$M = \begin{matrix} & \begin{matrix} c & d & a & b \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 1 \\ 0 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{bmatrix} \end{matrix}$$

Let $\rho_c = \lim_{n \rightarrow \infty} \phi^n(c)$. We can view ϕ as a substitution on the alphabet $\mathbb{A} = \{a, b, c, d\}$. Let \mathbb{A}_l be the set of words of length l on \mathbb{A} that appear in ρ_c . We want to calculate the frequency of occurrences of words, which cross c and d , in ρ_c . Let $w \in \mathbb{A}_l$ and let γ be a word of length l that starts with c . Using the same notation as in the previous example we have

$$\lim_{n \rightarrow \infty} \frac{(w, \phi^n(c))}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{M_l^n(w, \gamma)}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{\mathcal{B}_l^n(w, \gamma)}{\lambda^n} =: d_{w,c}$$

For example in length two we have $\mathbb{A}_2 = \{ab, ba, bb, ad, bd, ca, da, dc\}$ and $\mathcal{B}_2 = \{ad, bd, ca, da, dc\}$. We get the matrices

$$\mathcal{B}_1 = \begin{matrix} & \begin{matrix} b & c \end{matrix} \\ \begin{matrix} 1 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{matrix}, \quad \mathcal{B}_2 = \begin{matrix} & \begin{matrix} ca & da & dc & ad & bd \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

and compute the frequencies as in the previous example.

Example 5.3. Let $F_4 = \langle a, b, c, d \rangle$. Let G be the rose on four petals labeled a, b, c, d . Consider an outer automorphism Φ given by a train track representative $\phi : G \rightarrow G$ by

$$\phi(a) = ab, \phi(b) = bab, \phi(c) = cad, \phi(d) = dcad.$$

Let $\mathcal{A} = \{[\langle a, b \rangle]\}$. The transition matrix for ϕ is given by

$$M = \begin{matrix} & \begin{matrix} c & d & a & b \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 1 \\ 0 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{matrix}$$

Let $\rho_c = \lim_{n \rightarrow \infty} \phi^n(c)$. We can view ϕ as a substitution on the alphabet $\mathbb{A} = \{a, b, c, d\}$. As before we have

$$\lim_{n \rightarrow \infty} \frac{(w, \phi^n(c))}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{M_l^n(w, \gamma)}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{\mathcal{B}_l^n(w, \gamma)}{\lambda^n} =: d_{w,c}$$

where λ is the PF-eigenvalue of the top stratum. For length two we have $\mathbb{A}_2 = \{ab, ba, bb, ad, bd, ca, da, dc\}$ and $\mathcal{B}_2 = \{ad, bd, ca, da, dc\}$. We get the matrices

$$\mathcal{B}_1 = \begin{matrix} & \begin{matrix} b & c \end{matrix} \\ \begin{matrix} 1 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{matrix}, \quad \mathcal{B}_2 = \begin{matrix} & \begin{matrix} ca & da & dc & ad & bd \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

and compute the frequencies as above.

The next lemma defines a limiting current for a relative fully irreducible outer automorphism.

Lemma 5.4. *Let $\phi : G \rightarrow G$ be a completely split train track representative of Φ , a fully irreducible outer automorphism relative to \mathcal{A} . Let a be an edge in the top EG stratum H_r such that ρ_a is*

fixed under ϕ . Let v be any reduced edge path in G that crosses H_r . Let $d_{v,a}$ be the frequency of occurrence of v in ρ_a . Then the set of values

$$d_{v,a} + d_{\bar{v},a} =: \eta_\phi^a(v)$$

define a unique current η_ϕ^a relative to \mathcal{A} . That is,

- $\eta_\phi^a(v) \geq 0$,
- $\eta_\phi^a(v) = \eta_\phi^a(\bar{v})$,
- $\eta_\phi^a(v) = \sum_{e \in E} \eta_\phi^a(ve)$ where E is the set of edges of G and e is not equal to the inverse of the terminal edge of v .

For an edge $b \neq a$ in H_r we have that $\eta_\phi^b = \kappa \eta_\phi^a$ for some constant $\kappa(a, b, \phi|_{H_r})$. Thus for every fully irreducible outer automorphism relative to \mathcal{A} , we get a unique projective relative current, denoted $[\eta_\Phi^+] = [\eta_\phi^a]$.

Proof. The proof follows from Proposition 4.1 and Proposition 8.12 □

The projective relative current $[\eta_\Phi^+]$ is called the *stable current* for Φ . The stable current for ϕ^{-1} , denoted $[\eta_\Phi^-]$, is called the *unstable current* for Φ .

6. GOODNESS

In [BFH97], Bestvina, Feighn and Handel studied the legal structure of conjugacy classes under forward and backward iterates of a train track representative of a fully irreducible outer automorphism. In [Bri00], Brinkmann generalized some of those results to relative train track maps which we use in this section.

Throughout this section $\Phi \in \text{Out}(\mathbb{F}, \mathcal{A})$ will be a fully irreducible outer automorphism relative to \mathcal{A} and $\phi : G \rightarrow G$ a completely split train track representative of Φ with filtration $\emptyset = G_0 \subset G_1 \subset \dots \subset G_r = G$ such that $\mathcal{F}(G_{r-1}) = \mathcal{A}$, and H_r is the top EG stratum with PF eigenvalue $\lambda_\Phi > 1$. In this section we use Facts 2.1 about completely split train track maps.

In [Bri00], Brinkmann considers the following metric on G : edges in H_r get length according to the PF eigenvector such that the smallest length is one and hence edges in H_r get stretched by λ_Φ under the application of ϕ . Edges in G_{r-1} get edge length one.

Throughout we use the same notation for a conjugacy class in \mathbb{F} and its representative in G which is taken to be cyclically reduced. For a reduced path ρ in G by $[\phi(\rho)]$ we mean the tightened image of ρ . We define $i_r(\rho)$ to be the number of r -illegal turns in ρ , $l_r(\rho)$ the r -length of ρ and $L_r(\rho)$ the length of the longest r -legal segment in ρ . Recall from Section 2.7 that $L_r^c = \frac{2BCC(\phi)}{\lambda_\Phi - 1}$ is the critical r -length where $BCC(\phi)$ is the bounded cancellation constant.

We denote by ρ^{-k} a path in G with the property that the tightened image of $\phi^k(\rho^{-k})$ is ρ . For a subpath ρ of a path σ , let $[\phi^k(\rho)]_\sigma$ denote the maximal subpath of $[\phi^k(\rho)]$ contained in $[\phi^k(\sigma)]$.

The following proposition is a generalization of [BFH97, Lemma 2.9].

Proposition 6.1 ([Bri00, Lemma 6.2]). *Let $\phi : G \rightarrow G$ be a relative train track map and let H_r be an EG stratum. For every $L > 0$, $\exists M(L) > 0$ such that if ρ is a path in G_r that crosses H_r then one of the following holds:*

- (a) $[\phi^M(\rho)]$ contains an r -legal segment of r -length $> L$.
- (b) $[\phi^M(\rho)]$ has fewer r -illegal turns.
- (c) ρ can be expressed as a concatenation $\tau_1 \rho' \tau_2$, where $l_r(\tau_1) \leq 2L$, $l_r(\tau_2) \leq 2L$, $i_r(\tau_1) \leq 1$, $i_r(\tau_2) \leq 1$, and ρ' splits as a concatenation of pre-Nielsen paths (with one r -illegal turn each) and segments in G_{r-1} .

Lemma 6.2 (Backward iterations). *Let $\phi : G \rightarrow G$ be a completely split train track representative of a fully irreducible outer automorphism relative to \mathcal{A} . Given some number $L_0 > 0$, there exists $M > 0$, depending only on L_0 and H_r , such that for any subpath ρ of an \mathcal{A} -separable conjugacy class α realized in G_r with $1 \leq L_r(\rho) \leq L_0$ and $i_r(\rho) \geq 5$, we have*

$$\left(\frac{10}{9}\right)^n i_r(\rho) \leq i_r(\rho^{-nM})$$

for all $n > 0$.

Proof. In [Bri00, Lemma 6.4] Brinkmann proves the same statement for atoroidal outer automorphisms and for any path in G_r . The same proof follows by using Facts 2.1 about completely split train track representatives.

Given $L = L_0 + L_r^r$ choose M as in Proposition 6.1. Subdivide the path ρ into subpaths $\rho_1, \dots, \rho_m, \tau$ such that $i_r(\rho_i) = 5$ and $i_r(\tau) < 5$. Let ρ_i^{-M} be the pre-image of ρ_i under ϕ^M . Then ρ^{-M} is the concatenation of ρ_i^{-M} and τ^{-M} . We claim that $i_r(\rho_i^{-M}) \geq 6$ for all i . Suppose for contradiction that $i_r(\rho_i^{-M}) = 5$ for some i . Then by Proposition 6.1, ρ_i^{-M} splits as a concatenation of at least three pre-Nielsen paths and paths in G_{r-1} . By Facts 2.1 every Nielsen path has period one and there is at most one INP σ of height r . If σ is not closed then at least one end-point of σ is not contained in G_{r-1} . Therefore we cannot have three Nielsen paths in ρ_i^{-M} separated by paths in G_{r-1} . If σ is closed then its end point is not in G_{r-1} . Since α is \mathcal{A} -separable, it cannot have two consecutive occurrences of σ in it. Indeed, since σ (which is not contained in G_{r-1}) is fixed by ϕ it is not \mathcal{A} -separable. Therefore its relative Whitehead graph is connected without cut points. If α has two consecutive occurrences of σ then its relative Whitehead graph will also be connected without cut points but α is \mathcal{A} -separable. Therefore ρ and ρ_i^{-M} cannot have two consecutive occurrences of σ .

Thus $i_r(\rho^{-M}) \geq 6m + i_r(\tau) \geq (10/9)i_r(\rho)$ and the lemma follows by induction. \square

Lemma 6.3 ([Bri00, Lemma 6.5]). *Suppose H_r is an EG stratum. Given some $L > 0$, there exists some constant $C > 0$ such that for all paths $\rho \subset G_r$ with $1 \leq L_r(\rho) \leq L$ and $i_r(\rho) > 0$, we have*

$$C^{-1}i_r(\rho) \leq l_r(\rho) \leq Ci_r(\rho).$$

The notion of goodness was introduced in [Mar95] and formalized in [BFH97].

Definition 6.4 (Goodness). Given a loop or a path α in G_r that crosses H_r we say that the *good portion*, denoted g , of α is the set of r -legal segments that are r -distance L_r^c away from r -illegal turns. The *bad portion*, denoted b , is the part of α which is r -distance less than equal to L_r^c from an r -illegal turn. The r -length of α is equal to the r -length of g (denoted $g_r(\alpha)$) plus the r -length of b (denoted $b_r(\alpha)$). We define *goodness* of α as

$$\mathbf{g}(\alpha) = \frac{g_r(\alpha)}{l_r(\alpha)}.$$

Lemma 6.5. *Let $\delta > 0$ and $\epsilon > 0$ be given. Then there exists an integer $M = M(\delta, \epsilon)$ such that for any \mathcal{A} -separable conjugacy class α that crosses H_r with $\mathbf{g}(\alpha) \geq \delta$ we have $\mathbf{g}(\phi^m(\alpha)) \geq 1 - \epsilon$ for all $m \geq M$.*

The proof of the above lemma which is the same as in the absolute case can be found in [Uya14, Lemma 3.10].

Definition 6.6 (Desired growth [Bri00]). Let σ be a path in G that crosses an EG stratum H_r . We say σ has desired growth if there exist $N > 0, \lambda > 1, \epsilon > 0$ and a collection of subpaths S of σ such that the following hold:

(a) For every integer $n > 0$ and for every $\rho \in S$ we have

$$\lambda^n l_r(\rho) \leq \max\{l_r([\phi^{nN}(\rho)]_\sigma), l_r(\gamma)\},$$

where γ is a subpath of σ^{-nN} such that $[\phi^{nN}(\gamma)]_{\sigma^{-nN}} = \rho$.

(b) There is no overlap between distinct paths in S .

(c) The sum of the lengths of the paths in S is at least $\epsilon l_r(\sigma)$.

Lemma 6.7. *Let $\alpha \in \mathbb{F}$ be an \mathcal{A} -separable conjugacy class that crosses H_r . Then α has desired growth either under forward iteration or under backward iteration.*

Proof. Let $L_0 > L_r^c$ be a constant. There are several cases to consider.

- (1) $\frac{l_r(\alpha)}{i_r(\alpha)} \geq L_0$. The proof of [Bri00, Proposition 7.1 (2)(b)(i)] shows that in this case α has desired growth in the forward direction.
- (2) $\frac{l_r(\alpha)}{i_r(\alpha)} < L_0$.
 - (a) $i_r(\alpha) \geq 5$. By [Bri00, Proposition 7.1 (2)(b)(ii)] and using Lemma 6.2, 6.3 we get desired growth in the backward direction.
 - (b) $i_r(\alpha) < 5$. We have that α is \mathcal{A} -separable and crosses H_r non-trivially. Therefore α is not fixed and does not have two consecutive occurrences of a closed INP. Since $l_r(\alpha)$ is bounded from above, there are only finitely many possibilities for $\alpha \cap H_r$. Suppose the r -length of no segment of $\alpha \cap H_r$ grows under ϕ . Since there are only finitely many segments of H_r of bounded length, after passing to a power we can assume that a segment α_i of $\alpha \cap H_r$ is fixed under ϕ . Then the end points of α_i are in $H_r \cap G_{r-1}$. There has to be an illegal turn in α_i otherwise it would grow and in fact that illegal turn is an INP because it persists. But at least one end-point of an INP in G is not in G_{r-1} , thus we get a contradiction. Therefore we can pass to a uniform power M such that $\phi^M(\alpha)$ satisfies (1) and hence has desired growth in forward direction.

It can be seen in Brinkmann's proofs that the numbers N, λ, ϵ do not depend on a specific conjugacy class. \square

Let $\phi' : G' \rightarrow G'$ be a completely split train track representative of Φ^{-1} . Let $l_{r'}, i_{r'}, L_{r'}^c$ and C' be the corresponding notation related to ϕ' . There exists a constant B such that for any conjugacy class α we have

$$\frac{l_{r'}(\alpha)}{B} \leq l_r(\alpha) \leq B l_{r'}(\alpha).$$

Let \mathfrak{g}' denote the goodness with respect to the train track structure of ϕ' .

Lemma 6.8. *Given $\delta > 0$ there exists $M > 0$ such that for any \mathcal{A} -separable conjugacy class α that crosses H_r either*

- $\mathfrak{g}(\phi^{nM}(\alpha)) \geq \delta$ for all $n \geq 1$ or
- $\mathfrak{g}'((\phi')^{nM}(\alpha)) \geq \delta$ for all $n \geq 1$.

Proof. Let $L_0 > L_r^c$ be the constant from Lemma 6.7. By the same lemma there exist $N > 0, \lambda > 1$ and $\epsilon > 0$ such that any \mathcal{A} -separable conjugacy class that crosses H_r has desired growth. There are two cases:

- (a) Let's first consider the case when α has desired growth in the forward direction. This happens when $l_r(\alpha) \geq L_0 i_r(\alpha)$. For case 2(b) in the proof of Lemma 6.7 we pass to a uniform power of α which satisfies $l_r(\alpha) \geq L_0 i_r(\alpha)$. Let S be the collection of maximal r -legal subpaths of α of r -length at least $L_0 + 1$. Then by the choice of L_0 we have for $\rho \in S$,

$$l_r(\phi^{nN}(\rho)) \geq \lambda_\Phi^{nN} \frac{1}{L_0 + 1} l_r(\rho).$$

We have that the paths in S account for a definite fraction $\epsilon > 0$ of α . Now

$$g_r(\phi^{nN}(\alpha)) \geq \sum_{\rho \in S} [l_r(\phi^{nN}(\rho))]_{\alpha} \geq \sum_{\rho \in S} \lambda_{\Phi}^{nN} \frac{1}{L_0 + 1} [l_r(\rho)]_{\alpha} \geq \lambda_{\Phi}^{nN} \frac{1}{L_0 + 1} \epsilon l_r(\alpha).$$

We also have $l_r(\phi^{nN}(\alpha)) \leq \lambda_{\Phi}^{nN} l_r(\alpha)$. Thus we get

$$\mathfrak{g}(\phi^{nN}(\alpha)) \geq \frac{\epsilon}{L_0 + 1}.$$

(b) If α has desired growth in the backward direction then by Lemma 6.2 and Lemma 6.3 we have

$$Bl_{r'}((\phi')^{nN}(\alpha)) \geq l_r(\phi^{-nN}(\alpha)) \geq C^{-1} i_r(\phi^{-nN}(\alpha)) \geq \left(\frac{10}{9}\right)^n \frac{1}{C^2 B} l_{r'}(\alpha).$$

Now the number of r' -illegal turns in $(\phi')^{nN}(\alpha)$ is bounded above by those in α . We have

$$i_{r'}((\phi')^{nN}(\alpha)) \leq i_{r'}(\alpha) \leq C' l_{r'}(\alpha).$$

Also the bad portion of $(\phi')^{nN}(\alpha)$ is bounded from above by $2L_{r'}^c i_{r'}((\phi')^{nN}(\alpha))$. Thus

$$\mathfrak{g}'((\phi')^{nN}(\alpha)) \geq 1 - \frac{2L_{r'}^c C' B^2 C^2}{(10/9)^n} \geq 1 - \frac{2L_{r'}^c C' B^2 C^2}{(10/9)}.$$

Now by Lemma 6.5 we can find $M > 0$ such that either one of the goodness is greater than δ . \square

7. NORTH-SOUTH DYNAMICS

We are now ready to prove a north-south dynamic result. Recall we have Φ a fully irreducible outer automorphism relative to \mathcal{A} and a completely split train track representative $\phi : G \rightarrow G$. We also have a stable current $[\eta_{\Phi}^+]$ and an unstable current $[\eta_{\Phi}^-]$ in $\mathcal{MRC}(\mathcal{A})$.

Proposition 7.1. *Given a neighborhood U of $[\eta_{\Phi}^+]$ in $\mathcal{MRC}(\mathcal{A})$ there exists $0 < \delta < 1$ and $M(U) > 0$ such that for any $[\eta_{\alpha}] \in \mathcal{MRC}(\mathcal{A})$, with $\mathfrak{g}(\alpha) > \delta$ we have that $\phi^n([\eta_{\alpha}]) \in U$ for all $n \geq M$.*

The proof of the above lemma is similar to the proof of [Uya14, Lemma 3.11].

Lemma 7.2. *Given neighborhoods U and V of $[\eta_{\Phi}^+]$ and $[\eta_{\Phi}^-]$ in $\mathcal{MRC}(\mathcal{A})$ respectively there exists $M_1 > 0$ such that for any \mathcal{A} -separable conjugacy class α that crosses H_r either $\phi^m([\eta_{\alpha}]) \in U$ or $(\phi')^m([\eta_{\alpha}]) \in V$ for all $m \geq M_1$.*

The proof follows from Lemma 6.8 and Lemma 7.1.

Proposition 7.3 ([LU15, Proposition 3.4]). *Let $\phi : X \rightarrow X$ be a homeomorphism of X and assume that X is sufficiently separable, for example metrizable. Let $Y \subset X$ be a dense set, and let \mathcal{P}, \mathcal{Q} be two distinct ϕ -invariant points in X . Assume the following holds: for every neighborhood U of \mathcal{P} and V of \mathcal{Q} there exists an integer $M_2 \geq 1$ such that for all $m \geq M_2$ and all $y \in Y$ one has either $\phi^m(y) \in U$ or $\phi^{-m}(y) \in V$. Then ϕ^2 has uniform north-south dynamics from \mathcal{P} to \mathcal{Q} .*

Proposition 7.4 ([LU15, Proposition 3.5]). *Let $\phi : X \rightarrow X$ be as in Proposition 7.3 with distinct fixed points \mathcal{P} and \mathcal{Q} and assume that some power ϕ^s with $s \geq 1$ has uniform north-south dynamics from \mathcal{P} to \mathcal{Q} . Then ϕ also has uniform north-south dynamics from \mathcal{P} to \mathcal{Q} .*

Theorem A. *Let \mathcal{A} be a non-trivial free factor system of \mathbb{F} such that $\zeta(\mathcal{A}) \geq 3$. Let $\Phi \in \text{Out}(\mathbb{F}, \mathcal{A})$ be fully irreducible relative to \mathcal{A} . Then Φ acts with uniform north-south dynamics on $\mathcal{MRC}(\mathcal{A})$.*

Proof. The proof follows from Lemma 7.2, Proposition 7.3 and Proposition 7.4. \square

8. APPENDIX

8.1. Extension of relative currents. In this section we will prove Lemma 3.13, which says that given a relative current η_0 there exists a signed measured current η which is a k -extension of η_0 . We will first show that we can extend η_0 to a signed measured current η which may or may not be non-negative on all words of length less than equal to k . We then show how to modify η to get a k -extension of η_0 .

Remark 8.1. Throughout this section we will assume that \mathcal{A} has only one free factor A_0 . When \mathcal{A} has more than one free factor in it then the same process can be repeated for all the free factors independently of each other.

Notation:

- Let $\mathfrak{B}_{\mathcal{A}}$ be a relative basis of \mathbb{F} . Let s be the rank of the only free factor A_0 in \mathcal{A} . We will denote the generators of A_0 by a_i , $1 \leq i \leq s$. Also let $A := \{a_1^{\pm}, \dots, a_s^{\pm}\}$.
- Let S_k be the set of words in A_0 of length k with respect to $\mathfrak{B}_{\mathcal{A}}$. Let $\#S_k$ denote the cardinality of S_k which is $2s(2s-1)^{k-1}$.
- Let S_k^0 be a subset of S_k (chosen once and for all) such that for every $w \in S_k$ exactly one of w or \bar{w} appears in S_k^0 .
- We will use letters e, x, y, z to denote elements of $\mathfrak{B}_{\mathcal{A}}$.
- Whenever we write a forward (backward) extension of a word w by $e \in \mathfrak{B}_{\mathcal{A}}$ as we (ew) it is to be understood that e is not the inverse of the last (first) letter of w .

For every $k > 0$ we will define a signed measured current η on words in A_0 of length $(k-1)$ and use those values together with the additivity laws satisfied by η to define η on words of length k . To start with words of length one, choose arbitrary values for $\eta(a_i)$ for all $1 \leq i \leq s$. By induction assume we have values for $\eta(v)$ for all words v of length less than equal to $(k-1)$. By the additivity law satisfied by η there are some relations among the values $\eta(w)$ for $w \in S_k^0$ which need to be satisfied to extend η to length k words. We have for all $v \in S_{k-1}^0$

$$\begin{aligned}\eta(v) &= \sum_{e \in A} \eta(v e) + \sum_{e \notin A} \eta_0(v e), \\ \eta(\bar{v}) &= \sum_{e \in A} \eta(\bar{v} e) + \sum_{e \notin A} \eta_0(\bar{v} e).\end{aligned}$$

Since η is invariant under taking inverses the equation obtained from forward extension of \bar{v} is the same as the equation obtained from backward extension of v .

Rearranging the equations to have the unknown terms on left hand side we get

$$\begin{aligned}\sum_{e \in A} \eta(v e) &= \eta(v) - \sum_{e \notin A} \eta_0(v e) = c_v, \\ \sum_{e \in A} \eta(\bar{v} e) &= \eta(\bar{v}) - \sum_{e \notin A} \eta_0(\bar{v} e) = c_{\bar{v}}.\end{aligned}$$

Thus there are $\#S_{k-1}$ equations in $\#S_k^0$ variables and the number of variables are more than the number of equations. We denote this system of equations by E_{k-1}^1 , that is, equations obtained from relations among variables $\eta(w)$ for $w \in S_k^0$ coming from one edge extensions of length $(k-1)$ words. Similarly we can look at the system E_{k-1}^i .

Consider the augmented matrix $[M|c]$ for the system of equations E_{k-1}^1 with rows labeled by $v \in S_{k-1}$ and columns by $w \in S_k^0$. Then $M_{v,w} = 1$ if $w = ve$ or $\bar{w} = \bar{v}e$ for some $e \in A$ and 0 otherwise. We will denote a row vector of M by r_v corresponding to $v \in S_{k-1}$. We make some observations about the matrix M .

- Each column has exactly two ones. Indeed, $M_{v,w}$ is 1 exactly when v is a prefix of w or \bar{w} .

- There are $(2s-1)$ non-zero entries in each row because there are $(2s-1)$ possible extensions of v by $e \in A$.
- Any two distinct rows can be same in at most one column. Let w be common to two distinct rows r_{v_1} and r_{v_2} . Then

$$w = v_1 e_1 \text{ or } \overline{e_1} \overline{v_1} \quad \text{and} \quad w = v_2 e_2 \text{ or } \overline{e_2} \overline{v_2}$$

for some $e_1, e_2 \in A$. Then it must be true that v_1 begins with $\overline{e_2}$ and v_2 begins with $\overline{e_1}$. Thus w is uniquely determined.

Lemma 8.2. (a) For every $i \geq 1$ an equation in the system E_{k-i-1}^{i+1} is a linear combination of equations in the system E_{k-i}^i . Thus it is sufficient to look at the system E_{k-1}^1 to get all constraints satisfied by $\eta(w)$ for all $w \in S_k^0$.

(b) Let $u \in S_{k-2}$. Then we have

$$\sum_{x \in A} r_{xu} = \sum_{x \in A} r_{x\overline{u}}.$$

(c) The set of relations $\sum_{x \in A} r_{xu} = \sum_{x \in A} r_{x\overline{u}}$ for every $u \in S_{k-2}$ generate any other relation among the rows of M .

(d) We also have that

$$\sum_{x \in A} c_{xu} = \sum_{x \in A} c_{x\overline{u}}$$

where c_v is the constant term of the equation determined by $v \in S_{k-1}$.

(e) The system of equations E_{k-1}^1 is consistent and hence has a solution. Thus we can define η on words of length k .

Proof. (a) Let $u \in S_{k-i-1}$. Then

$$\eta(u) = \sum_{x \in A} \eta(ux) + \sum_{x \notin A} \eta(ux).$$

By equations in E_{k-i}^i we have

$$\eta(ux) = \sum_{y \in \mathbb{F}, |y|=i} \eta(uxy).$$

Adding all these equations over $x \in \mathfrak{B}_A$ we get

$$\eta(u) = \sum_{x, y \in \mathbb{F}, |x|=1, |y|=i} \eta(uxy) = \sum_{z \in \mathbb{F}, |z|=i+1} \eta(uz)$$

Thus we recovered an equation in E_{k-i-1}^{i+1} by a combination of equations in E_{k-i}^i .

(b) For every $x \in A$, $M_{xu,w} \neq 0$ exactly when $w = xu\overline{y}$ or $w = y\overline{u}x$ for some $y \in A$. Therefore if $M_{xu,w} \neq 0$ then $M_{y\overline{u},w} \neq 0$ for some $y \in A$.

(c) Consider a minimal relation R given by $\sum_{v \in S_{k-1}} d_v r_v = 0$ where $d_v \in \mathbb{R}$. We can rescale the

equation such that coefficient of at least one row, say r_{xu} for some $x \in A$ and $u \in S_{k-2}$, is 1.

For every $y \in A$ and $w = xu\overline{y}$ we have $M_{xu,w} = M_{y\overline{u},w} = 1$. Thus r_{xu} and $r_{y\overline{u}}$ share exactly one common entry w and no other row has a non-zero entry in w . Thus $d_{y\overline{u}} = -1$. Now consider $y \in A$. For any $z \in A$ and $w = y\overline{u}z$ we have $M_{y\overline{u},w} = M_{\overline{z}u,w} = 1$. Thus $d_{\overline{z}u} = 1$.

Hence our minimal relation is just $\sum_{x \in A} r_{xu} - \sum_{y \in A} r_{y\overline{u}} = 0$.

(d) We have

$$\begin{aligned}
\sum_{x \in A} c_{xu} &= \sum_{x \in A} \eta(xu) - \sum_{x \in A, y \notin A} \eta(xuy) \\
&= \eta(u) - \sum_{x \notin A} \eta(xu) - \sum_{x \in A, y \notin A} \eta(xuy) \\
&= \eta(u) - \sum_{x \notin A, y \in \mathfrak{B}_A} \eta(xuy) - \sum_{x \in A, y \notin A} \eta(xuy)
\end{aligned}$$

and similarly

$$\begin{aligned}
\sum_{x \in A} c_{x\bar{u}} &= \eta(u) - \sum_{x \notin A, y \in \mathfrak{B}_A} \eta(x\bar{u}y) - \sum_{x \in A, y \notin A} \eta(x\bar{u}y) \\
&= \eta(u) - \sum_{x \notin A, y \in \mathfrak{B}_A} \eta(\bar{y}u\bar{x}) - \sum_{x \in A, y \notin A} \eta(\bar{y}u\bar{x})
\end{aligned}$$

We see that

$$\sum_{x \notin A, y \in \mathfrak{B}_A} \eta(xuy) + \sum_{x \in A, y \notin A} \eta(xuy) = \sum_{x \notin A, y \in \mathfrak{B}_A} \eta(\bar{y}u\bar{x}) + \sum_{x \in A, y \notin A} \eta(\bar{y}u\bar{x})$$

as individual terms can be made equal. Geometrically, we are looking at the same subset of $\partial^2 \mathbb{F}$ as a union of cylinder sets in two different ways. See Figure 1 when $\mathbb{F} = \langle a, b, c, d \rangle$.

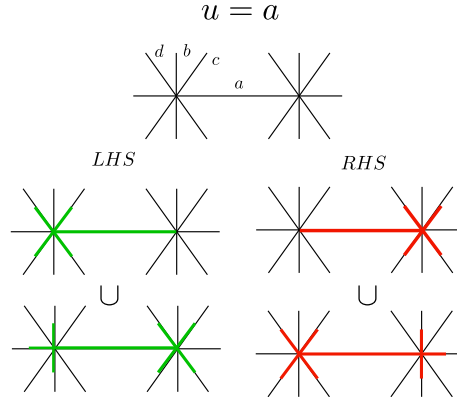


FIGURE 1.

(e) Since the relations which generate all other relations among the rows of M are consistent we get that $[M|c]$ has a solution. \square

Proof of Lemma 3.13. Given a relative current η_0 , by Lemma 8.2 we can find a signed measured current η such that $\eta_0(w) = \eta(w)$ for all $w \in \mathbb{F} \setminus \mathcal{A}$. This extension need not be non-negative on all words of length less than equal to k . Let $-M$ for $M > 0$ be the smallest value attained by $\eta(w)$ for a word $w \in \mathcal{A}$ with $|w| \leq k$. Consider a signed measured current $\eta_{\mathcal{A}, C}$ defined as follows:

$$\eta_{\mathcal{A}, C}(w) = \frac{C}{(2s-1)^{|w|-1}} \text{ for } w \in \mathcal{A} \text{ and } 0 \text{ otherwise.}$$

For $C = M(2s-1)^{k-1}$, $\eta + \eta_{\mathcal{A}, C}$ is non-negative on words of length less than equal to k . \square

8.2. Substitution Dynamics. Let \mathbb{A} be a finite set with cardinality greater than equal to two. Let ζ be a substitution on \mathbb{A} , that is, a map from \mathbb{A} to the set of non-empty words on \mathbb{A} which associates to a letter $e \in \mathbb{A}$ the word $\zeta(e)$ with length $|\zeta(e)|$. The substitution ζ induces a map on the set of all words on \mathbb{A} by concatenation, that is,

$$\zeta(x_1x_2 \dots x_m) = \zeta(x_1)\zeta(x_2) \dots \zeta(x_m)$$

where $x_1x_2 \dots x_m$ is a word on \mathbb{A} . Thus we can define iterates ζ^n for all $n \geq 1$. To the substitution ζ we associate its transition matrix, denoted M , where for $a, b \in \mathbb{A}$, $M(a, b)$ is the number of occurrence of a in $\zeta(b)$. The transition matrix for ζ^n is given by M^n . Likewise, we define a map from $\mathbb{A}^{\mathbb{N}}$ to $\mathbb{A}^{\mathbb{N}}$, the set of all infinite words on \mathbb{A} , also denoted ζ , by the formula $\zeta(x_1x_2 \dots) = \zeta(x_1)\zeta(x_2) \dots$.

Suppose ζ admits a fixed point, denoted $\rho \in \mathbb{A}^{\mathbb{N}}$, such that $\zeta^k(\rho) = \rho$ for all $k \geq 1$. From now on we only keep in the alphabet \mathbb{A} the letters that actually appear in ρ .

For every $l > 0$ let \mathbb{A}_l denote the set of all words on \mathbb{A} of length l that appear in ρ . We define a substitution ζ_l on \mathbb{A}_l as follows: let $w = x_1x_2 \dots x_l \in \mathbb{A}_l$. We define $\zeta_l(w) := w_1w_2 \dots w_{|\zeta(x_1)|}$ where $w_i \in \mathbb{A}_l$ and w_i is the length l subword of $\zeta(w)$ starting at the i^{th} position of $\zeta(x_1)$. In other words $\zeta_l(w)$ consists of the ordered list of the first $|\zeta(x_1)|$ subwords of length l of the word $\zeta(w)$. The substitution ζ_l extends to a map on the set of all words on \mathbb{A}_l . We denote by $|\cdot|_l$ the length of words on \mathbb{A}_l . We have $|\zeta_l(w)|_l = |\zeta(x_1)|$. We denote by M_l the transition matrix for ζ_l . It is clear from definitions that $(\zeta^n)_l = (\zeta_l)^n$.

A substitution is called *irreducible* if for every pair $a, b \in \mathbb{A}$ there exists $k := k(a, b)$ such that a occurs in $\zeta^k(b)$. A substitution is called *primitive* if there exists k such that for every pair $a, b \in \mathbb{A}$, a occurs in $\zeta^k(b)$.

We are interested in understanding the frequency of occurrence of words on \mathbb{A} that occur in a fixed point ρ . In [Que87] a theory for understanding these frequencies for a primitive substitution is developed. We want to generalize the theory of primitive substitutions to a substitutions which may not be primitive but are primitive on a subset of the alphabet.

8.2.1. Eigenvalues for M and M_l . The main result from this section is Proposition 8.5. Consider an alphabet $\mathbb{A} = \bigsqcup_{i=0}^k B_i$. We define a *partial order on the alphabet* as follows. First we define a partial order on subsets of \mathbb{A} given by $B_i > B_j$ for $i < j$. For example $B_0 > B_1$ and so on. Thus we get a partial ordering on the letters of \mathbb{A} where $a > b$ if $a \in B_i$ and $b \in B_j$ where $i < j$. The alphabet \mathbb{A}_l can now be given a partial lexicographic order as well. We will consider a substitution ζ on \mathbb{A} with the following properties:

- For $a \in B_i$, $\zeta(a)$ contains letters only from B_j for $j \geq i$. This implies that the transition matrix M for ζ is lower triangular block diagonal with respect to the partial order on the set $\{B_i\}_{i=0}^k$. We will denote the diagonal blocks of M also by B_i for $0 \leq i \leq k$ where B_0 is the top left block, followed by B_1 and so on.
- If B_i is a primitive block then $\zeta(a)$ for $a \in B_i$ ends and begins in a letter in B_i .
- B_0 is primitive.

Lemma 8.3. *Let B_i be a primitive block of M . After possibly passing to a power of ζ , there exists $a \in B_i$ such that $\zeta(a)$ begins in a . Also $\rho_a := \lim_{n \rightarrow \infty} \zeta^n(a)$ is fixed by ζ , that is, $\zeta(\rho_a) = \rho_a$. If $b \in B_i$ is another letter which begins in b and ρ_b is fixed by ζ then the set of subwords of ρ_a and ρ_b are the same.*

Proof. Consider a function $f : B_i \rightarrow B_i$ where for $a \in B_i$ $f(a)$ is the first letter of $\zeta(a)$. Since B_i is a finite set some power of f has a fixed point. After passing to a power in needed let $a \in B_i$ be a fixed point of f . Since $\zeta(a)$ begins with a , we have that $\zeta^n(a)$ begins with $\zeta^{n-1}(a)$ for every $n > 0$. Thus ρ_a is fixed by ζ . Since B_i is a primitive block $\zeta^m(a)$ contains b for some $n > 0$. Thus subwords that appear in ρ_b also appear in ρ_a and vice versa. \square

We say a word w on \mathbb{A} *crosses* B_i if w contains a letter in B_i . We are interested in understanding the frequency of occurrence of words in $\rho := \rho_0$ which cross B_0 .

Example 8.4. Let $\mathbb{A} = \{a, b, c, d\}$. Let ζ be given as $\zeta(a) = abbab, \zeta(b) = bababbab, \zeta(c) = cad, \zeta(d) = dcad$. The transition matrix for ζ and ζ_2 are given by

$$M = \begin{matrix} & \begin{matrix} c & d & a & b \end{matrix} \\ \begin{matrix} c & d & a & b \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{bmatrix} \end{matrix}, \quad M_2 = \begin{matrix} & \begin{matrix} ca & da & dc & ad & bd & ab & ba & bb \end{matrix} \\ \begin{matrix} ca & da & dc & ad & bd & ab & ba & bb \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 \end{bmatrix} \end{matrix}.$$

We now want to understand the spectrum of M_l .

Proposition 8.5. *For every $l \geq 2$, the eigenvalues of M_l are those of M with possibly some additional eigenvalues of absolute value less than equal to one.*

The three lemmas that follow will be used to prove Proposition 8.5. Since $(\zeta^n)_l = (\zeta_l)^n$ we have $(M^n)_l = (M_l)^n$, which we now denote by M_l^n unless we need to specify the order. We denote the rows and columns of M by R_x and C_x for $x \in \mathbb{A}$, those of M_l by R_w and C_w and those of M_l^n by $R_{n,w}$ and $C_{n,w}$ for $w \in \mathbb{A}_l$.

Lemma 8.6. *Let $n \geq 2$. Let M, M_l, M_l^n be transition matrices for $\zeta, \zeta_l, \zeta_l^n$ respectively. Then*

- (a) M_l is a lower triangular block diagonal matrix with respect to the partial order on \mathbb{A}_l .
- (b) Let $w \in \mathbb{A}_l$ start with $x \in \mathbb{A}$. Then the sum of the entries of C_w is the same as the sum of the entries of C_x which is equal to $|\zeta(x)|$.
- (c) Let $w_1, w_2 \in \mathbb{A}_l$ be such that both words begin with $x \in \mathbb{A}$. Then the entries of C_{w_1} and C_{w_2} differ at most by $(l-1)$. The entries of C_{n,w_1} and C_{n,w_2} also differ at most by $(l-1)$.

Proof. (a) Clear from definitions of M and M_l .

(b) Let w, x be as in the statement of the lemma. Then $|\zeta_l(w)|_l = |\zeta(x)|$ which implies that column sum of C_w is same as that of C_x .

(c) Let w_1, w_2, x be as in the statement of the lemma. Then $\zeta_l(w_1)$ and $\zeta_l(w_2)$ differ only when the length l words starting at some position in $\zeta(x)$ are not subwords of $\zeta(x)$. If $|\zeta(x)| \geq l$ then the first time such a word occurs is when it starts at position $(l-1)$ from the end of $\zeta(x)$. If $|\zeta(x)| < l$ then $\zeta_l(w_1)$ and $\zeta_l(w_2)$ can differ in at most $|\zeta(x)| < l$ length l words. Thus there are at most $(l-1)$ such words. Replace ζ, ζ_l by $\zeta^n, (\zeta^n)_l$ above to conclude that entries of C_{n,w_1} and C_{n,w_2} also differ at most by $(l-1)$. \square

Lemma 8.7. *If Q is a $s \times s$ matrix such that absolute values of all its entries are bounded above by $\delta > 0$ then the absolute values of the eigenvalues of Q are bounded above by $s\delta$.*

Proof. Let $\lambda \neq 0$ be an eigenvalue of Q and let $v = (v_1, \dots, v_s)$ be a corresponding eigenvector. Let r_i denote rows of Q . Then $|r_i \cdot v| = |\lambda v_i|$ which gives $|\lambda v_i| \leq \delta \sum_{j=1}^s |v_j|$ for every $1 \leq i \leq s$. Adding all the inequalities together we get $|\lambda| \leq s\delta$. \square

For every $B_i \subset \mathbb{A}$ let $\widetilde{B_i} \subset \mathbb{A}_l$ be the set of all words w that start with a letter in B_i and w does not cross B_j for any $j < i$. For every $B_i \subset \mathbb{A}$ let $\overline{B_i} \subset \mathbb{A}_l$ be the set of all words w that start with a letter in B_i and there exists a $j < i$ such that w crosses B_j (note that $\overline{B_0}$ is empty). Then we have that $\widetilde{B_i} \cup \overline{B_i}$ is the union of all words of length l that start with a letter in B_i . The partial order on \mathbb{A}_l defined earlier gives that $\widetilde{B_0} > \overline{B_1} > \widetilde{B_1} > \dots > \overline{B_k} > \widetilde{B_k}$. The matrix M_l is lower

triangular block diagonal with respect to this partial order on \mathbb{A}_l . For a subset $S \subset \mathbb{A}_l$ we denote by S the transition matrix of ζ_l restricted to S .

- Lemma 8.8.** (a) For every $0 \leq i \leq k$, the characteristic polynomial of B_i divides the characteristic polynomial of \widetilde{B}_i .
 (b) The eigenvalues of \widetilde{B}_i are those of B_i with possibly some additional eigenvalues of absolute value less than equal to one.
 (c) The eigenvalues of \overline{B}_i have absolute value less than equal to one.

Proof. (a) Consider the matrix $P_i = \widetilde{B}_i - \lambda I$. We will do certain row and column operations on this matrix to reduce it to a lower triangular block diagonal matrix with $B_i - \lambda I$ as a diagonal block, which would imply that the characteristic polynomial of B_i divides the characteristic polynomial of \widetilde{B}_i . For later use we denote the other diagonal block of P_i by Q .

We first perform the following row operations: for every $x \in B_i$ choose a word $w \in \widetilde{B}_i$ such that w starts with x . For every such w replace the row R_w of \widetilde{B}_i by the sum of rows R_u for all $u \in \widetilde{B}_i$ that start with x . Rearrange the rows and columns such that top left block is indexed by the chosen words w . We call the rearranged matrix P'_i . The top left block of P'_i is exactly $B_i - \lambda I$. Indeed, suppose $w, u \in \widetilde{B}_i$ in the top left block of P'_i start with $x, y \in B_i$ respectively. Then $P'_i(w, v)$ is exactly the number of occurrences of y in $\zeta(x)$.

Now for any two columns C_{w_1} and C_{w_2} of P'_i , where w_1, w_2 start with the same letter in B_i , the first few entries (as many as the number of rows in the top left block of P'_i) are equal. Now we perform column operations as follows: for every $x \in B_i$ and w the chosen word in the top left block we subtract C_w from C_u for every $u \neq w$ that start with x . Thus we have a lower triangular block diagonal matrix, again denoted P'_i , with diagonal blocks $B_i - \lambda I$ and Q .

- (b) Consider the lower block diagonal matrix P'_i from above. Eigenvalues of P'_i not coming from the block $B_i - \lambda I$ come from the lower block, denoted Q . By Lemma 8.6(c) the entries of Q are bounded in absolute value by $(l-1)$. We claim that the eigenvalues of Q are bounded in absolute value by one.

Let λ_0 be an eigenvalue of Q and hence of \widetilde{B}_i . Then for $n \geq 1$, λ_0^n is an eigenvalue of $(\widetilde{B}_i)^n$ which is a diagonal block of $(M_l)^n = (M^n)_l$. Thus λ_0^n is an eigenvalue of $(\widetilde{B}_i)^n$ that does not come from eigenvalue of B_i^n , the corresponding diagonal block of M^n . Applying part (b) to ζ^n we get that $(\widetilde{B}_i)^n$ can also be put in a lower triangular block diagonal form with diagonal blocks $B_i^n - \lambda I$ and Q' whose entries are bounded by $(l-1)$ and hence every eigenvalue bounded in absolute value by size of Q' times $(l-1)$ by Lemma 8.7. Thus $|\lambda_0^n|$ is uniformly bounded which can happen only when $|\lambda_0| \leq 1$.

Thus all eigenvalues of \widetilde{B}_i are eigenvalues of B_i with the exception of some eigenvalues whose absolute value is less than equal to one.

- (c) Let λ be an eigenvalue of \overline{B}_i . Then λ^n is an eigenvalue of $(\overline{B}_i)^n$, the diagonal block of $(M^n)_l$ corresponding to words that start with a letter in B_i and there exists a $j < i$ such that they cross B_j . For every n , the entries of $(\overline{B}_i)^n$ are bounded by $(l-1)$. Indeed, if w is a length l word that starts with x then only the words that start at some position less than l away from the last letter of $\zeta^n(x)$ belong to $(\overline{B}_i)^n$. This implies that eigenvalues of $(\overline{B}_i)^n$ are uniformly bounded. That is $|\lambda^n|$ is uniformly bounded which can happen only when $|\lambda| \leq 1$. \square

Proof of Proposition 8.5. Since eigenvalues of a lower triangular block diagonal matrix are obtained from eigenvalues of each block the proposition follows from Lemma 8.8. \square

8.2.2. Frequency of words. Recall that we want to understand the frequency of occurrence of words which cross B_0 in ρ . Let λ be the top eigenvalue of the block B_0 of M . Consider a subset $\mathcal{B}_l := \widetilde{B}_0 \cup (\bigcup_{i=1}^k \overline{B}_i)$ of \mathbb{A}_l . Then the set of all length l words that cross B_0 is a subset of \mathcal{B}_l . The

transition matrix of ζ_l restricted to \mathcal{B}_l is also lower triangular block diagonal with respect to the order $\overline{B_0} > \overline{B_1} > \dots > \overline{B_k}$ of words in \mathcal{B}_l . Then by Lemma 8.8, $\lambda > 1$ is the top eigenvalue of \mathcal{B}_l with multiplicity one. Since \mathcal{B}_l is a diagonal block of M_l we have $M_l^n(w, \alpha) = \mathcal{B}_l^n(w, \alpha)$ for all $w, \alpha \in \mathbb{A}_l$ that cross B_0 .

For w, v words on \mathbb{A} or \mathbb{A}_l let (w, v) denote the number of occurrences of w in v .

Lemma 8.9. *Let $a \in B_0$ and let $\rho_a = \lim_{n \rightarrow \infty} \zeta^n(a)$ be such that $\zeta(\rho_a) = \rho_a$. Let $w \in \mathbb{A}_l$ be a word that crosses B_0 . Then*

$$\text{frequency of occurrence of } w \text{ in } \rho_a = \lim_{n \rightarrow \infty} \frac{(w, \zeta^n(a))}{\lambda^n} =: d_{w,a}$$

exists and is non-negative. Here λ is the top eigenvalue of B_0 .

Proof. Let $\alpha \in \mathbb{A}_l$ start with a . For n large the number of occurrences of w in $\zeta^n(a)$ is approximately the same as number of occurrences of w in $\zeta_l^n(\alpha)$. Also

$$(w, \zeta_l^n(\alpha)) = M_l^n(w, \alpha).$$

We have

$$\lim_{n \rightarrow \infty} \frac{(w, \zeta^n(a))}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{(w, \zeta_l^n(\alpha))}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{M_l^n(w, \alpha)}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{\mathcal{B}_l^n(w, \alpha)}{\lambda^n} =: d_{w,a}.$$

Indeed, the limit exists because λ is the top eigenvalue of \mathcal{B}_l . The limit is non-negative because it is a sequence of non-negative numbers. The limit does not depend on the exact choice of α because by Lemma 8.6(c), any two columns of M_l^n starting with the same letter in \mathbb{A} differ by a bounded amount and thus give the same limit. \square

Lemma 8.10 (Kirchhoff's Law). *Let $a \in B_0$. Let $w \in \mathbb{A}_l$ cross B_0 . Let we and ew be length one extensions of w by $e \in \mathbb{A}$. Then*

$$d_{w,a} = \sum_{e \in \mathbb{A}} d_{we,a} = \sum_{e \in \mathbb{A}} d_{ew,a}.$$

Proof. We have $(w, \zeta^n(a))$ and $\sum_{e \in \mathbb{A}} (we, \zeta^n(a))$ differ only when $\zeta^n(a)$ ends in w so the difference is at most one. Thus

$$\left| \frac{(w, \zeta^n(a))}{\lambda^n} - \sum_{e \in \mathbb{A}} \frac{(we, \zeta^n(a))}{\lambda^n} \right| \rightarrow 0 \text{ as } n \rightarrow \infty$$

which implies that $d_{w,a} = \sum_{e \in \mathbb{A}} d_{we,a}$. Similarly $d_{w,a} = \sum_{e \in \mathbb{A}} d_{ew,a}$. \square

Lemma 8.11. *Let $a, b \in B_0$ be distinct. Then*

$$d_{w,b} = \kappa d_{w,a}$$

for every word w that crosses B_0 where $\kappa = \kappa(a, b, \zeta|_{B_0})$.

Proof. Let's first consider the case when length of w is one. We have that ζ restricted to B_0 is primitive with top eigenvalue $\lambda > 1$. Then

$$d_{w,a} = \lim_{n \rightarrow \infty} \frac{M^n(w, a)}{\lambda^n} = \lim_{n \rightarrow \infty} \frac{B_0^n(w, a)}{\lambda^n}.$$

Since B_0 is primitive the limit of B_0^n/λ^n is a matrix P that is spanned by a positive eigenvector corresponding to λ . Since left eigenvector of B_0 is also positive we have that all columns of P are positive multiples of each other. Thus $d_{w,b} = P(w, b)$ is a scalar multiple of $d_{w,a} = P(w, a)$ which does not depend on w . We call this constant κ_1 .

Now consider the case when length of w is l . We will first show that the constant κ_l , where $d_{w,b} = \kappa_l d_{w,a}$, does not depend on w and then we will show that $\kappa_l = \kappa_1$ for all $l \geq 2$. Since λ is the unique top eigenvalue of \mathcal{B}_l , $\lim_{n \rightarrow \infty} \mathcal{B}_l^n/\lambda^n$ is a matrix P whose column span is an eigenvector

corresponding to λ . Thus $d_{w,b} = P(w,b)$ is a scalar multiple of $d_{w,a} = P(w,a)$ which does not depend on w . We call this constant κ_l .

Now we will show that $\kappa_l = \kappa_1$. Let w be a word of length one. We have $d_{w,b} = \sum_{e \in \mathbb{A}} d_{we,b}$. Also $d_{w,b} = \kappa_1 d_{w,a}$ and $d_{we,b} = \kappa_2 d_{we,a}$. Thus we have $\kappa_1 d_{w,a} = \kappa_2 \sum_{e \in \mathbb{A}} d_{we,a} = \kappa_2 d_{w,a}$ which implies $\kappa_2 = \kappa_1$. We can repeat the same argument to get $\kappa_l = \kappa_1$ for every $l \geq 2$. \square

To summarize the results about substitutions we have the following proposition.

Proposition 8.12. *Let ζ be a substitution on an alphabet \mathbb{A} such that the transition matrix is lower triangular block diagonal with top left block B_0 primitive, and for every $e \in B_0$, $\zeta(e)$ starts and ends with a letter in B_0 . Then there is a fixed infinite word ρ obtained by iterating a letter in B_0 under ζ . Moreover, the frequency of a word w on \mathbb{A} in ρ that crosses B_0 is well defined up to scale and satisfies Kirchhoff's law.*

8.2.3. Train track map as a substitution. Let Φ be a free group outer automorphism. Let $\phi : G \rightarrow G$ be a completely split train track representative of Φ with filtration $\emptyset = G_0 \subset G_1 \subset \dots \subset G_K = G$. The transition matrix for ϕ , denoted M_ϕ , is lower triangular block diagonal. Let a be an edge in an EG stratum H_r such that up to taking powers $\phi(a)$ starts with a . Let $\rho_a = \lim_{n \rightarrow \infty} \phi^n(a)$. We want to understand the frequency of occurrence of paths in G_r that cross H_r and appear in ρ_a . We may not be able to treat ϕ as a substitution directly since there could be cancellations and inverse of edges would have to be treated separately. The proof of the next proposition explains how to view a completely split train track map as a substitution for the purpose of calculating frequencies of certain paths.

We set up some notation about exceptional paths that will be used in the next proposition. Let $e_1, e_2 \in G$ be two linear edges such that $\phi(e_1) = e_1 \sigma^{d_1}$ and $\phi(e_2) = e_2 \sigma^{d_2}$ where σ is an INP and $d_1 \neq d_2$. If $d_1, d_2 > 0$ then $x_m = e_1 \sigma^m e_2$ where $m \in \mathbb{Z}$ is an exceptional path. We say x_m has height $|m|$. Let $\delta = d_1 - d_2$. Then $\phi(x_m)$ is the exceptional path $x_{m+\delta}$.

Proposition 8.13. *Let $\phi : G \rightarrow G$ be a completely split train track map. Let a be an edge in an EG stratum H_r such that $\phi(a)$ starts with a , and let $\rho_a := \lim_{n \rightarrow \infty} \phi^n(a)$. Let γ be a path in G_r that crosses H_r . Then*

$$\lim_{n \rightarrow \infty} \frac{(\gamma, \phi^n(a))}{\lambda^n} =: d_{\gamma,a}$$

exists and is non-negative. Here λ is the Perron-Frobenius eigenvalue of the aperiodic EG stratum H_r . If $b \in H_r$ is another edge then for every γ as above,

$$d_{\gamma,b} = \kappa d_{\gamma,a}$$

where κ is a constant with $\kappa = \kappa(a,b,\phi|_{H_r})$.

Proof. The ray ρ_a is completely split and the terms of the complete splitting, called splitting units, of ρ_a form an alphabet \mathbb{A}_∞ for a substitution, but \mathbb{A}_∞ can be infinite if there are exceptional paths. We will define a finite alphabet \mathbb{A}_γ , which depends on γ , by identifying some elements in \mathbb{A}_∞ in order to calculate the frequency of occurrence of γ in ρ_a . We will also show that the frequency of γ in ρ_a does not depend on the choice of the alphabet \mathbb{A}_γ . Let \mathcal{N} be the set of all INPs, r -taken connecting paths and exceptional paths that appear in ρ_a .

Before we define the alphabet \mathbb{A}_γ , we define a relation from the set of all finite paths in ρ_a that cross H_r , denoted $\mathcal{P}_r(\rho_a)$, to the set of all finite words on \mathbb{A}_∞ , denoted $\mathcal{W}(\mathcal{A}_\infty)$,

$$r : \mathcal{P}_r(\rho_a) \rightarrow \mathcal{W}(\mathcal{A}_\infty).$$

For a finite path $\gamma \in \mathcal{P}_r(\rho_a)$, the set $r(\gamma)$ consists of the following words:

- (a) If an occurrence of γ in ρ_a is a concatenation of splitting units then $r(\gamma)$ contains the corresponding word on \mathbb{A}_∞ .

- (b) If an occurrence of γ in ρ_a is a subword of an INP σ then $r(\gamma)$ contains the element of \mathbb{A}_∞ determined by σ , denoted w_σ . There are only finitely many INPs that appear in ρ_a therefore the number of occurrences of a path γ in an INP is bounded. If σ contains n occurrences of γ then we let $r(\gamma)$ contain n copies of w_σ . Note that a path γ in $\mathcal{P}_r(\rho_a)$ is not contained in an exceptional path or an r -taken connected path.
- (c) If an occurrence of γ has partial overlaps with some elements of \mathcal{N} then consider a path γ' such that γ' is the smallest subpath of ρ_a that is a concatenation of splitting units and which contains γ . Then $r(\gamma)$ contains the word on \mathbb{A}_∞ corresponding to γ' .

Thus every occurrence of γ in ρ_a corresponds to the occurrence of some word from $r(\gamma)$ in ρ_a . Note that $r(\gamma)$ can be an infinite set, for instance, when γ has partial overlap with infinitely many exceptional paths in ρ_a . But the set of words in $r(\gamma)$ viewed in the alphabet \mathbb{A}_γ , defined below, will form a finite set. We now define the alphabet \mathbb{A}_γ . For simplicity, let's assume that γ intersects only one family of exceptional paths, say determined by linear edges $e_1, e_2 \in G$.

- Let $\mathcal{H} = \{H_r = H_{i_1}, \dots, H_{i_k}\}$ be the collection of strata crossed by edges in H_r . For every H_{i_j} , we let $\mathbb{A}(H_{i_j})$ be the alphabet which contains an edge and its inverse as distinct letters if they both appear in ρ_a otherwise the edge with the orientation that appears.
An edge in G is called a *Type 1* edge if it always appears with positive or negative orientation but not both in ρ_a . An edge which appears with both orientations in ρ_a is said to be of *Type 2*. If H_{i_j} is an EG stratum then either all edges in H_{i_j} are Type 1 or all are Type 2 (see [Uya14] for proof). Thus if we consider a substitution on $\mathbb{A}(H_r)$ representing ϕ restricted to H_r then the substitution will be primitive.
- Now consider splitting units which are INPs, r -taken connecting paths and exceptional paths. Let $\mathbb{A}(\mathcal{N}_\gamma)$ be an alphabet defined as follows:
 - (a) $\mathbb{A}(\mathcal{N}_\gamma)$ contains oriented INPs and r -taken connecting paths that appear in ρ_a . In general there can be infinitely many INPs in G_r but only finitely many appear in ρ_a .
 - (b) When γ contains an exceptional path determined by e_1, e_2 or a subsegment of an exceptional path determined by e_1, e_2 : let N be the maximum length of such an exceptional path that appears in γ , in $\phi(e)$ for all edges e in H_r and in an r -taken connecting path. Then $\mathbb{A}(\mathcal{N}_\gamma)$ contains exceptional paths determined by e_1, e_2 of height less than equal to $N + 1$ as distinct elements. All other exceptional paths determined by e_1, e_2 of length greater than $N + 1$ correspond to a single element of $\mathbb{A}(\mathcal{N}_\gamma)$.
 - (c) When γ does not intersect an exceptional path determined by e_1, e_2 : then all exceptional paths determined by e_1, e_2 correspond to a single element of $\mathbb{A}(\mathcal{N}_\gamma)$.
- Let \mathbb{A}_γ be defined as the set $\mathbb{A}(H_{i_1}) \sqcup \dots \sqcup \mathbb{A}(H_{i_k}) \sqcup \mathbb{A}(\mathcal{N}_\gamma)$ and let $\zeta_{\gamma, \phi}$ be a substitution on \mathbb{A}_γ determined by ϕ . Let $\tilde{r}(\gamma)$ be the set of words in $r(\gamma)$ viewed in the alphabet \mathbb{A}_γ . Then $\tilde{r}(\gamma)$ is a finite set of words on \mathbb{A}_γ . The frequency of occurrence of a path $\gamma \in \mathcal{P}_r(\rho_a)$ in ρ_a is given by the sum of the frequencies of the words in $\tilde{r}(\gamma)$. \square

If we replace $N + 1$ by $N + C$ for any $C \geq 1$ in the above construction to get a different alphabet \mathbb{A}'_γ then the frequency of γ calculated from the two alphabets is the same. More precisely, let \mathbb{A}_γ and \mathbb{A}'_γ be two alphabets which differ only in the naming of exceptional paths determined by e_1, e_2 of length greater than $N + 1$. Let ζ and ζ' be the corresponding substitutions, and let $\tilde{r}(\gamma)$ and $\tilde{r}'(\gamma)$ be the set of words in $r(\gamma)$ viewed in \mathbb{A}_γ and \mathbb{A}'_γ respectively. An exceptional path maps to another exceptional path under ϕ . Therefore ζ and ζ' have the same growth rate when restricted to $\mathbb{A}(H_r)$. Since the number of occurrences of γ does not change, we get that the two substitutions yield the same frequency for words in $\tilde{r}(\gamma)$ and $\tilde{r}'(\gamma)$ and hence the same frequency for γ .

Remark 8.14. Different substitutions constructed in the previous proposition for different words γ differ only in exceptional paths. Since an exceptional path maps to another exceptional path these different substitutions have the same growth rate when restricted to $\mathbb{A}(H_r)$. Also Kirchhoff's

law still holds for frequencies of paths in ρ_a because $(\gamma, \phi^n(a))$ and $\sum_{e \in G_r} (\gamma e, \phi^n(a))$ differ by a bounded amount.

We do some examples below to exhibit how to view a completely split train track map as a substitution.

Example 8.15. Let R_3 be the rose on three petals with labels a, b, c . Consider a homotopy equivalence $\phi : R_3 \rightarrow R_3$ given by

$$\phi(a) = a, \phi(b) = Bac, \phi(c) = CBac.$$

Here capital letters denote inverses. The transition matrix for ϕ is

$$\begin{array}{c} b \quad c \quad a \\ \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

There are two stratum $H_1 = \{a\}$ and $H_2 = \{b, c\}$. Every edge in H_2 is of Type 2. Let $\rho_C = \lim_{n \rightarrow \infty} \phi^n(C)$. We have $\mathcal{H} = \{H_2, H_1\}$, $\mathbb{A}(H_2) = \{b, c, B, C\}$ and $\mathbb{A}(H_1) = \{a, A\}$. Since there are no exceptional paths we use one alphabet $\mathbb{A} = \{b, c, B, C, a, A\}$ and a substitution ζ_ϕ on \mathbb{A} whose transition matrix is given by

$$\begin{array}{c} b \quad c \quad B \quad C \quad a \quad A \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Example 8.16. Consider a homotopy equivalence $\phi : R_5 \rightarrow R_5$ given by

$$\phi(a) = ab, \phi(b) = bab, \phi(c) = cae, \phi(d) = dc\sigma d, \phi(e) = dcae$$

where $\sigma = abAB$ is a Nielsen path. There are two stratum $H_1 = \{a, b\}$ and $H_2 = \{c, d, e\}$. Let $\rho_c = \lim_{n \rightarrow \infty} \phi^n(c)$. We have $\mathcal{H} = \{H_2, H_1\}$, $\mathbb{A}(H_2) = \{c, d, e\}$, $\mathbb{A}(H_1) = \{a, b\}$ and $\mathbb{A}(\mathcal{N}) = \{\sigma\}$. Since there are no exceptional paths we use one alphabet $\mathbb{A} = \{c, d, e, a, b, \sigma\}$ and a substitution ζ_ϕ on \mathbb{A} whose transition matrix is given by

$$\begin{array}{c} c \quad d \quad e \quad a \quad b \quad \sigma \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

In this example the frequency of occurrence of the edge path ca in ρ_c comes from the occurrence of the words ca and $c\sigma$ in $\rho_c(\zeta_\phi)$. Thus the frequency of ca in ρ_c is equal to $d_{ca,c} + d_{c\sigma,c}$.

Example 8.17. This example illustrates the discussion of exceptional paths in Proposition 8.13. Consider a homotopy equivalence $\phi : R_6 \rightarrow R_6$ given by

$$\begin{aligned} \phi(a) &= ab, & \phi(b) &= bab, \\ \phi(c) &= c\sigma^2, & \phi(d) &= d\sigma, \\ \phi(e) &= eaf, & \phi(f) &= f\sigma Deaf, \end{aligned}$$

where $\sigma = abAB$. Some exceptional paths are $x_i = c\sigma^i D$ for $i > 0$. To calculate the frequency of words like fx_4 or $f\sigma^4$ in ρ_f we consider the alphabet $\mathbb{A} = \{e, f, a, b, c, D, x_1, x_2, x_3, x_4, x_5, \sigma, \bar{\sigma}\}$ and substitution ζ such that

$$\begin{aligned}
\zeta(a) &= ab, & \zeta(b) &= bab, \\
\zeta(c) &= c\sigma^2, & \zeta(d) &= d\sigma, \\
\zeta(f) &= fx_1eaf, & \zeta(e) &= eaf, \\
\zeta(\sigma) &= \sigma, & \zeta(\bar{\sigma}) &= \bar{\sigma}, \\
\zeta(x_i) &= x_{i+1} & & \text{for } 1 \leq i \leq 3 \\
\zeta(x_4) &= \zeta(x_5) = x_5,
\end{aligned}$$

The path $\gamma = fc\sigma^4$ does not occur as a concatenation of splitting units in ρ_f . The path $\gamma' = fx_4$ is the smallest subpath of ρ_f that is a concatenation of splitting units and contains γ . Thus the frequency of occurrence of γ is the same as the frequency of occurrence of γ' .

8.3. Example of an outer automorphism relative to \mathcal{A} when $\text{rank}(\mathcal{A}) = \text{rank}(\mathbb{F})$. This just an example of a relative fully irreducible outer automorphism when rank of cofactor of \mathcal{A} is zero. Let $\mathbb{F} = \langle a, b, c \rangle$ and let $\mathcal{A} = \{[\langle a \rangle], [\langle b \rangle], [\langle c \rangle]\}$. Let Φ be an outer automorphism given by

$$\Phi(a) = a, \Phi(b) = aCbca, \Phi(c) = CbcBc.$$

Let $\phi : G \rightarrow G$ be a relative train track representative of Φ with G as in Figure 2.

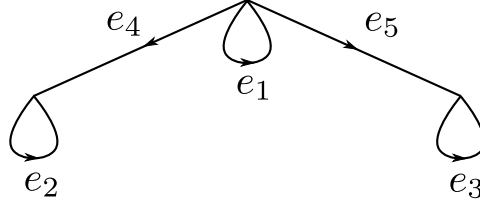


FIGURE 2. The graph G

with marking given by

$$a \rightarrow e_1, b \rightarrow e_1e_4e_2E_4E_1, c \rightarrow e_5e_3E_5.$$

The map ϕ is given as follows

$$\begin{aligned}
\phi(e_1) &= e_1 & \phi(e_2) &= e_2 & \phi(e_3) &= e_3 \\
\phi(e_4) &= e_5E_3E_5e_1e_4 & \phi(e_5) &= e_5E_3E_5e_1e_4e_2E_4E_1e_5
\end{aligned}$$

and the transition matrix for ϕ is

$$\begin{bmatrix}
e_5 & e_4 & e_3 & e_2 & e_1 \\
3 & 2 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 & 1
\end{bmatrix}.$$

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